CLUSTER SETS ON OPEN RIEMANN SURFACES

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ABSTRACT. Generalizations of theorems of Gross-Iversen type on exceptional values are given for analytic mappings on open Riemann surfaces.

The classical, well-known theorems of Picard and Iversen (cf. [2, p. 3]) concerning isolated, essential singularities were extended to those of Hällström and Cartwright (cf. [2, pp. 10, 15]) concerning essential singularities of capacity zero, respectively. These extensions were furthermore extended to the theorems of Tsuji and Noshiro (cf. [2, pp. 14, 19]) which are stated as follows:

Let $D$ be an arbitrary domain, $B$ its boundary, $A$ a compact set of capacity zero on $B$ and $z_0$ a point of $A$. Let $\varphi(z)$ be single-valued and meromorphic in $D$. $C_D(\varphi, z_0)$ and $C_{B-A}(\varphi, z_0)$ denote the full cluster set of $\varphi(z)$ at $z_0$ and the boundary cluster set of $\varphi(z)$ at $z_0$ modulo $A$, respectively.

1. Every value of $C_D(\varphi, z_0) - C_{B-A}(\varphi, z_0)$ is assumed by $\varphi(z)$ infinitely often in any neighborhood of $z_0$ except for a possible set of values of capacity zero.

2. If $\alpha \in C_D(\varphi, z_0) - C_{B-A}(\varphi, z_0)$ is an exceptional value of $\varphi(z)$ in a neighborhood of $z_0$, then either $\alpha$ is an asymptotic value of $\varphi(z)$ at $z_0$ or there is a sequence $\xi_n \in A$ $(n = 1, 2, 3, \ldots)$ converging to $z_0$ such that $\alpha$ is an asymptotic value of $\varphi(z)$ at each $\xi_n$.

In this paper, (1) and (2) will be generalized for analytic mappings from open Riemann surfaces into Riemann surfaces. These generalizations will be given by Theorem 1 and Theorem 2.

Let $f$ be an analytic mapping from an open Riemann surface $R$ into a Riemann surface $S$. Let $R^*$ and $S^*$ denote a metrizable compactification and an arbitrary compactification of $R$ and $S$, respectively. $\bar{X}$ and $\text{bdy } X$ mean the closure and the boundary of a subset $X$ of $R^*$ or $S^*$ with respect to $R^*$ or $S^*$, respectively. $\partial X$ means the relative boundary of a subset $X$ of $R$ or $S$ with respect to $R$ or $S$.

We write $\Delta = R^* - R$. The full cluster set of $f$ at $p \in \Delta$ is defined as

$C(f, p) = \bigcap_{r > 0} (f(U(p, r) \cap R))$, where $U(p, r)$ denotes the $r$-neighborhood of $p$. For a set $E$ on $\Delta$, the boundary cluster set of $f$ at $p$ modulo $E$ is defined...
as \( C_{\Delta -E}(f, p) = \bigcap_{r>0} \bigcup_{q \in W(p, r)} \mathcal{C}(f, q) \), where \( W(p, r) = U(p, r) \cap \Delta - E - \{p\} \). It is said that a path \( \gamma(t) (0 < t < 1) \) in \( R \) tends to a connected set \( K (\subset \Delta) \), when for any \( r \)-neighborhood \( U(K, r) \) of \( K \), there is a \( t(K, r) \) such that \( \gamma(t) \subset U(K, r) \) for all \( t > t(K, r) \). Henceforth let \( V(P), V_0(P) \) and \( V^*(P) \) denote parametric disks about a point \( P \) of \( R \) or \( S \).

**Theorem 1.** Let \( E \) be a polar set on \( \Delta \) and \( p \) a point of \( E \). Let \( r_1 > r_2 > \cdots > r_n \to 0 \). If \( E \cap \text{bdy } U(p, r_n) = \emptyset \) for every \( n \), then every point of \( F = C(f, p) \cap S - C_{\Delta -E}(f, p) \) is assumed by \( f \) infinitely often in any \( U \in \{U(p, r)\} \), with a possible exceptional set of capacity zero.

**Proof.** Let \( f_{U \cap R} \) denote the restriction of \( f \) to \( U \cap R \) and \( n(f_{U \cap R}, b) \) the number of the points of \( f_{U \cap R}^{-1}(b) \) for each \( b \in S \), where each point is counted with its multiplicity. \( n(f_{U \cap R}, b) \) is lower semicontinuous on \( S \) and, hence, \( F_n = \{b \in S; n(f_{U \cap R}, b) < n\} (n = 0, 1, 2, \ldots) \) is relatively closed in the open set \( S - C_{\Delta -E}(f, p) \). Suppose that \( \{b \in F_n; n(f_{U \cap R}, b) < \infty\} \) is of positive capacity. Then there is an \( N (0 < N < \infty) \) for which \( F_{N-1} = \emptyset \) is of capacity zero and \( F_N \) is of positive capacity, where \( F_{-1} = \emptyset \). It is possible to find some \( c \in F_N - F_{N-1} \), which is not a branch point, such that for any \( V(c), V(c) \cap F_N \) is of positive capacity.

First consider the case where \( U(p, r) \cap (\Delta - E) \neq \emptyset \) for every \( U(p, r) \). Then \( C_{\Delta -E}(f, p) \neq \emptyset \). Choose an open set \( G (\not\subset \Delta) \) containing \( C_{\Delta -E}(f, p) \). There are \( V(a_i) \subset (\subset U) (i = 1, 2, \ldots, N) \) such that \( f(a_i) = c \) and \( V(a_i) \cap V(a_j) = \emptyset (i \neq j) \), and which are mapped onto a \( V_0(c) \), satisfying \( V_0(c) \cap G = \emptyset \), by \( f \) homeomorphically. There is a \( U' \in \{U(p, r)\} \) such that \( U' \subset U \) and \( \bigcup_{q \in W} \mathcal{C}(f, q) \subset G \), where \( W = U' \cap \Delta - E \). For each \( q \in W \), there is a \( U(q) \in \{U(q, r)\} \) satisfying \( f(U(q) \cap R) \subset G \). Since \( f(\bigcup_{q \in W} (U(q) \cap R)) \cap V_0(c) = \emptyset \), \( \bigcup_{q \in W} (U(q) \cap R) \cap V_0(c) = \emptyset \) and, hence, \( f^{-1}(V_0(c)) \cap W = \emptyset \). Therefore \( f^{-1}(V_0(c)) \cap U' \cap \Delta = \emptyset \).

Choose a \( U(p, r_{N*}) \) such that \( \overline{U(p, r_{N*})} \subset U' \) and \( U(p, r_{N*}) \cap V(a_i) = \emptyset (i = 1, 2, \ldots, N) \). \( f(U(p, r_{N*}) \cap R) \) contains \( c \), because \( c \in C(f, p) \). Since \( E \cap \text{bdy } U(p, r_{N*}) = \emptyset \), it is easy to see that \( f^{-1}(V_0(c)) \cap \text{bdy } U(p, r_{N*}) \) is compact in \( R \). Hence \( f(R \cap \text{bdy } U(p, r_{N*})) (\not\subset V_0(c) \) is relatively closed on \( V_0(c) \). Thus it is possible to choose some \( V^*(c) \) satisfying \( V^*(c) \cap f(R \cap \text{bdy } U(p, r_{N*})) = \emptyset \).

Take a component \( \mathcal{D}^* (\subset U(p, r_{N*})) \) of \( f^{-1}(V^*(c)) \). \( \partial f(\mathcal{D}^*) - \partial V^*(c) \) has regular points relative to the Dirichlet problem (cf. [1, pp. 42, 50]). Let \( h \) be a continuous function with the property that \( h \) is equal to 0 on \( \partial f(\mathcal{D}^*) \cap \partial V^*(c) \) and \( 0 < h < 1 \) in \( \partial f(\mathcal{D}^*) - \partial V^*(c) \). Let \( u \) be the solution of the Dirichlet problem in \( f(\mathcal{D}^*) \) with \( h \) as its boundary function. Then \( 0 < u < 1 \) in \( f(\mathcal{D}^*) \) and \( \lim_{q \to a} u \circ f(\mathcal{D}^*) = 0 \) at every \( q \in \partial f(\mathcal{D}^*) \). Since \( E \) is polar, there is a positive superharmonic function \( s \) on \( R \) with \( \lim_{q \to a} s(q) = 0 \) at every \( q \in \partial f(\mathcal{D}^*) \cup (\mathcal{D}^* \cap \Delta) \). It follows from the minimum principle (cf. [1, p. 11]) that \( u \circ f_{D^*} > 0 \) in \( D^* \). This implies a contradiction.
Next consider the case where $U(p, r) \cap (\Delta - E) = \emptyset$ for some $U(p, r)$. Then $C_{\Delta - E} = \emptyset$. We have a contradiction, as we see easily from the above proof. Thus the proof of Theorem 1 is complete.

**Theorem 2.** If, under the hypotheses of Theorem 1, $e \in F$ is an exceptional point of $f$ in some $U^* \in \{U(p, r)\}$, then either $e$ is an asymptotic point of $f$ at $p$ or there is an infinite sequence of connected sets $K_n (\subset E)$ converging to $p$ such that $e$ is the asymptotic point of $f$ along a path tending to $K_n$.

**Proof.** First consider the case where $U(p, r) \cap (\Delta - E) \neq \emptyset$ for every $U(p, r)$. Take any $U'' \in \{U(p, r)\}$ contained in $U^*$. Since $n(f_{U'^{n}}(p, e)) = 0$, it is possible to take $U'$, $U(p, r_{n^*})$, and $V^*(e)$ in the proof of Theorem 1 such that $U'(p, r_{n^*}) \subset U' \cap U''$ and $V^*(e) \cap f(R \cap bdy U(p, r_{n^*})) = \emptyset$. Any component $D^* (\subset U(p, r_{n^*}))$ of $f^{-1}(V^*(e))$ is not relatively compact in $R$.

Let $g_{V^*(e)}(b, e)$ denote the Green's function for $V^*(e)$ with pole at $e$. Suppose that $f(D^*) \not\ni e$. Then $g_{V^*(e)}(f_{D^*}(a), e)$ is bounded on $D^*$. As in the proof of Theorem 1, it follows that $-g_{V^*(e)}(f_{D^*}(a), e) > 0$ in $D^*$. This is impossible. Therefore $f(D^*) \ni e$.

Let $w = \psi(b)$ be a local parameter of $V^*(e)$, and write $\psi(V^*(e)) = \{w; |w| < 1\} (\psi(e) = 0)$ and $W_{1/n} = \{w; |w| < 1/n\} (n = 1, 2, 3, \ldots)$. Let $\{D_n\}$ be an infinite sequence of components of $f^{-1} \circ \psi^{-1}(W_{1/n})$ such that $D_{n+1} \subset D_n \subset D^*$, and $\{p_n\}$ an infinite sequence of points $p_n \in D_n$. $D_n$ is not relatively compact in $R$ and $f(D_n) \ni e$. For any compact set $K' (\subset R)$, there is an $N_0$ such that $D_n \subset R - K'$ for all $n > N_0$. Furthermore, as in the proof of Theorem 1, $D_n \subset E$. For each $n$, there is a simple arc $\lambda_n$ joining $p_n$ to $p_{n+1}$ in $D_n$ such that $f(\lambda_n) \subset \psi^{-1}(W_{1/n})$. Thus the path $\lambda = \cup \lambda_n$ tends to a component of $E \cap U(p, r_{n^*})$ and has the property that $f \rightarrow e$ along $\lambda$. Since $E \cap bdy U(p, r_n) = \emptyset$ for every $n$, our assertion is proved.

Next consider the case where $U(p, r) \cap (\Delta - E) = \emptyset$ for some $U(p, r)$. From the above proof, we see easily that our conclusion holds. Thus the proof of Theorem 2 is complete.

**References**


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