CONDITIONS FOR GENERATING
A NONVANISHING BOUNDED ANALYTIC FUNCTION

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Abstract. B. A. Taylor and L. A. Rubel have posed the problem of finding necessary and sufficient conditions on a set of given functions \( f_1, f_2, \ldots, f_n \) in \( H^\infty \) such that there exist functions \( g_1, g_2, \ldots, g_n \) in \( H^\infty \) with \( \sum_{i=1}^{n} f_i g_i \neq 0 \) in the open unit disc. L. A. Rubel has conjectured that a necessary and sufficient condition is that there exist a harmonic minorant of \( \log |\sum_{i=1}^{n} |f_i|| \) in the open unit disc. The major result of this paper proves that the conjecture is true if one of the given functions \( f_1, f_2, \ldots, f_n \) has a zero set which is an interpolation set for \( H^\infty \).

Let \( D \) be the open unit disc in the complex plane and let \( H^\infty \) denote the space of bounded holomorphic functions on \( D \). If \( f_1, f_2, \ldots, f_n \) are \( n \) given functions in \( H^\infty \), we seek necessary and sufficient conditions that there are functions \( g_1, g_2, \ldots, g_n \) in \( H^\infty \) with

\[
\sum f_i(z) g_i(z) \neq 0, \quad z \in D. \tag{1}
\]

If it happens that \( |\sum f_i g_i| \) is actually bounded away from zero on \( D \), then \( f_1, f_2, \ldots, f_n \) generate \( H^\infty \) (as an ideal); it is a known and difficult theorem of L. Carleson that \( f_1, f_2, \ldots, f_n \) generate \( H^\infty \) if and only if

\[
\sum |f_i(z)| > \delta > 0, \quad z \in D \tag{2}
\]

(see [2, p. 163]). Since we ask here for only the weaker condition that \( \sum f_i g_i \) does not vanish in \( D \), it is to be expected that (2) will be replaced by some weaker condition. L. A. Rubel has conjectured that (1) holds for some \( g_1, g_2, \ldots, g_n \) if and only if the function

\[
\mu(z) = \log \sum |f_i(z)| \tag{3}
\]

has a harmonic minorant on \( D \). We show in what follows that this conjecture is true under the additional hypothesis that the zero set of some \( f_j \) is an interpolation sequence for \( H^\infty \).

A sequence of points \( \{z_k\}_{k=1}^{\infty} \) in \( D \) is called an interpolation sequence for \( H^\infty \) if, for each bounded sequence of complex numbers \( \{w_k\}_{k=1}^{\infty} \), there exists a function \( f \) in \( H^\infty \) such that \( f(z_k) = w_k \) for every \( k \). A Blaschke product is an

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\(^1\)All summations will be indexed from \( i = 1 \) to \( i = n \) unless otherwise indicated.

\(^2\)Private communications.

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analytic function $B$ of the form

$$B(z) = z^n \prod_{k=1}^{\infty} \left[ \frac{\alpha_k \cdot (\alpha_k - z)}{|\alpha_k| \cdot (1 - \alpha_k z)} \right]^{p_k}$$

where

(i) $p, p_1, p_2, \ldots$ are nonnegative integers;
(ii) the $\alpha_k$ are distinct nonzero numbers in $D$;
(iii) the product $\prod_{k=1}^{\infty} |\alpha_k|^{p_k}$ is convergent.

For further discussion on interpolation sequences and Blaschke products, see [2, pp. 66–74, 194–207] and [1, pp. 18–29, 136–143].

Suppose $\Sigma f_i g_i \neq 0$ in $D$ where the $f_i$'s and $g_i$'s are in $H^\infty$. Then $\Sigma f_i g_i = e^h$ where $h$ is analytic in $D$.

Consequently, $e^{Re h} < c \Sigma |f_i|$ and so $Re h - \log c < \log \Sigma |f_i|$ and $Re h - \log c$ is harmonic. This shows that the condition that $\log \Sigma |f_i|$ have a harmonic minorant is necessary. It is clearly sufficient for the case $n = 1$. For $n > 2$, suppose the zero set of some $f_i$ is finite. If $\mu(z)$ has a harmonic minorant, then the zero sets of $f_1, f_2, \ldots, f_n$ are mutually disjoint. Since the only possible limit points of the zero sets lie on the unit circle, there exists an open set $V$ containing the zero set of $f_i$ such that $|f_i(z)| > \delta_1 > 0$ outside $V$ and $\Sigma_{k \neq j} |f_k| > \delta_2 > 0$ on $V$. Consequently, $\Sigma |f_i| > \delta > 0$ on the open disc, and by the result of L. Carleson cited above, $f_1, f_2, \ldots, f_n$ generate $H^\infty$. Hence we now restrict our attention to the case where $n > 2$ and the zero sets of the $f_i$'s are all infinite.

**Theorem.** Given $f_1, f_2, \ldots, f_n$ in $H^\infty$ with the zero set of $f_1$ an interpolation sequence for $H^\infty$, there exist functions $g_1, g_2, \ldots, g_n$ in $H^\infty$ with $\Sigma f_i g_i \neq 0$ in $D$ if and only if $\log \Sigma |f_i|$ has a harmonic minorant.

We need a preliminary result:

**Lemma.** Let $B$ be a Blaschke product whose zero set $\{z_k\}_{k=1}^{\infty}$ is an interpolation sequence for $H^\infty$. Let $f_1, f_2, \ldots, f_n$ be functions in $H^\infty$. Then $I_1 = I_2$ where

$$I_1 = \{ Bg_0 + \sum f_i g_i: g_0, g_1, g_2, \ldots, g_n \text{ are in } H^\infty \}$$

and

$$I_2 = \{ g \in H^\infty: |g(z_k)|\left(\sum |f_i(z_k)|\right)^{-1} < M \text{ for all } k \text{ where } M \text{ depends on } g \}.$$

**Proof of Lemma.** Let $F = Bg_0 + \sum f_i g_i$ lie in $I_1$. Then

$$|B(z_k)g_0(z_k) + \sum f_i(z_k)g_i(z_k)|\left(\sum |f_i(z_k)|\right)^{-1} = \left|\sum f_i(z_k)g_i(z_k)\left(\sum |f_i(z_k)|\right)^{-1}\right| < \|g\|_\infty$$

for all $k$. Hence $I_1 \subset I_2$. 

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Now suppose \( g \in I_2 \). For \( i = 1, 2, \ldots, n \) and \( k = 1, 2, 3, \ldots \), define \( \alpha_{ik} \) by

\[
\alpha_{ik} = \begin{cases} 
0 & \text{if } f_i(z_k) = 0, \\
\text{sgn } f_i(z_k) & \text{if } f_i(z_k) \neq 0.
\end{cases}
\]

Then \( \{\alpha_{ik}\}_{k=1}^\infty \in l^\infty \) for \( i = 1, 2, \ldots, n \). Since \( \{z_k\}_{k=1}^\infty \) is interpolating, there exist \( g_1, g_2, \ldots, g_n \) in \( H^\infty \) such that \( g_i(z_k) = \alpha_{ik} \) for \( i = 1, 2, \ldots, n \) and \( k = 1, 2, 3, \ldots \). Consequently,

\[
|g(z_k)\left(\sum f_i(z_k)g_i(z_k)\right)^{-1}| = |g(z_k)|\left(\sum |f_i(z_k)|\right)^{-1} \leq M
\]

for all \( k \). So \( \{g(z_k)(\sum f_i(z_k)g_i(z_k))^{-1}\}_{k=1}^\infty \in l^\infty \), and there exists \( h \in H^\infty \) such that

\[
h(z_k) = g(z_k)\left(\sum f_i(z_k)g_i(z_k)\right)^{-1}
\]

for every \( k \). We have that \( g - h\sum f_ig_i = 0 \) on \( \{z_k\}_{k=1}^\infty \) which implies that \( g - h\sum f_ig_i = Bh_1 \), where \( h_1 \in H^\infty \). Thus \( g = Bh_1 + \sum f_ig_ih \) and so \( g \in I_1 \). This shows that \( I_2 \subset I_1 \), and the proof is complete.

**Proof of Theorem.** Let \( B_1 \) be the Blaschke factor of \( f_1 \). Suppose \( \log |S|f_i| \) has harmonic minorant \( \mu \). Let \( v \) be a harmonic conjugate of \( \mu \). Then \( \Sigma |f_i| > e^\mu = |e^{\mu+i\nu}| \) and thus \( e^{\mu+i\nu} \in H^\infty \). Let \( h = \mu + iv \) and let \( \{z_k\}_{k=1}^\infty \) be the zero set of \( B_1 \). Then \( |e^{h(z_k)}(\Sigma |f_i(z_k)|)|^{-1} \leq 1 \) for all \( k \). By the Lemma, there exist functions \( g_0, g_1, \ldots, g_n \) in \( H^\infty \) such that \( B_0g_0 + \sum f_ig_i = e^h \). Now \( f_0 = B_1G_1 \), where \( G_1 \in H^\infty \) and \( G_1 \neq 0 \) in \( D \). Hence \( B_1G_1g_0 + \sum f_ig_iG_1 = G_1e^h \), and the proof is complete.

The condition in the Theorem that \( \{z_k\}_{k=1}^\infty \) be an interpolation sequence can be weakened so that \( \{z_k\}_{k=1}^\infty \) is the union of a finite number of interpolation sequences. The argument is by induction on the number of interpolation sequences and is straightforward. P. J. McKenna has announced the following related result: Let \( S = \{z_n\} \) be a sequence of points in the open unit disc satisfying the Blaschke condition. Let \( \mu \) be the discrete measure concentrated on the sequence \( S \), with weights \( \mu(\{z_n\}) = 1 - |z_n|^2 \), \( n = 1, 2, 3, \ldots \). Then \( \mu \) is a Carleson measure if and only if \( S \) is a finite union of interpolating sequences.

**References**


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