PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 66, Number 1, September 1977

THE HYPERBOLICITY OF THE COMPLEMENT OF $2n+1$ HYPERPLANES IN GENERAL POSITION IN $\mathbf{P}_n$, AND RELATED RESULTS

MARK L. GREEN[1]

ABSTRACT. Using a modified version of a technique of R. Brody, a simple proof is found that the complement of $2n+1$ hyperplanes in general position in $\mathbf{P}_n$ is complete hyperbolic and hyperbolically embedded in $\mathbf{P}_n$. In fact, a more general result is obtained showing that a suitable Picard theorem is sufficient to imply hyperbolicity in a large class of algebro-geometric situations.

1. **Introduction**. Our main theorem, first proved by A. Bloch [1] (see also [3]) in 1926, is the following:

THEOREM 1. *The complement of* $2n+1$ *hyperplanes in general position in* $\mathbf{P}_n$ *is complete hyperbolic and hyperbolically embedded in* $\mathbf{P}_n$.

(For the meaning of these terms, see [6] and [9].)

Bloch's proof is exceedingly complex, and there has been considerable interest in finding a simpler argument. A differential-geometric argument of M. Cowen [4] gives a nice proof for $n=2$, but breaks down for higher n. The method of proof here will be a noncompact version of a simple but fertile technique introduced by R. Brody [2]. A similar proof has been found independently by A. Howard.

We will obtain the more general result:

THEOREM 2. *Let* D *be a union of (possibly singular) hypersurfaces* $D_1, \ldots, D_m$ *in a compact complex manifold* V. *Then* $V-D$ *is complete hyperbolic and hyperbolically embedded in* V *provided*

(1) *there are no nonconstant holomorphic maps* $C \to V-D$, *and*

(2) *there are no nonconstant holomorphic maps* $C \to D_{i_1} \cap \cdots \cap D_{i_k} - (D_{j_1} \cup \cdots \cup D_{j_l})$, *for any choice of distinct indices so* $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{1, \ldots, m\}$.

Received by the editors June 11, 1976 and, in revised form, September 10, 1976, November 23, 1976, and January 17, 1977.

AMS (MOS) subject classifications (1970). Primary 30A70, 32H25; Secondary 32H20.

Key words and phrases. Hyperbolic manifold, holomorphic curve, value distribution theory, Picard theorem, hyperplanes in general position, complex projective spaces.

[1]The author gratefully acknowledges support of the N.S.F.

In fact, we need only assume there are no nonconstant holomorphic maps of finite order $\leqslant 2$ *in* (1) *and* (2).

(For the meaning of finite order, see [7].)

Some applications of Theorem 2 other than $\mathbf{P}_n$ minus hyperplanes will be discussed in §4.

2. **Brody's reparametrization lemma.** Let M be a complex manifold, possibly with boundary. Let H be a continuous differential metric, i.e. a continuous function $|\ |_H$ on the tangent bundle of M satisfying $|\alpha v_x|_H = |\alpha|\,|v_x|_H$ for all $\alpha \in \mathbf{C}$, $v_x \in T_x(M)$. Let $\Delta(r)$ be the disc of radius r and

$$\eta_r(z) = r^2/\left(r^2 - |z|^2\right).$$

BRODY'S REPARAMETRIZATION LEMMA. *Let* $\Delta(r) \to^f M$ *be holomorphic, with* $|df(0)|_H > c$. *Then there exists* $\Delta(r) \to^g M$ *holomorphic so*

(1) $|dg(0)|_H = c$,

(2) $|dg(z)|_H/\eta_r(z) \leqslant c$, *all* $z \in \Delta(r)$,

(3) Image$(g) \subset$ Image(f).

PROOF. Let $\Delta(r) \to^{f_t} M$ be given by $f_t(z) = f(tz)$. Then

$$\frac{|df_t(z)|_H}{\eta_r(z)} = t\,\frac{r^2 - |z|^2}{r^2 - |tz|^2} \cdot \frac{|df(tz)|_H}{\eta_r(tz)}.$$

If $u(t) = \sup_{z \in \Delta(r)} |df_t(z)|_H/\eta_r(z)$, then:

(1) u is finite, monotone increasing, and continuous on $[0, 1)$.

(2) $c < u(1) \leqslant \infty$. Thus, for suitable $t \in [0, 1)$, we have $u(t) = c$. The maximum occurs at a point $z_0 \in \Delta(r)$, and if $h(z) = (z - z_0)/(1 - \bar{z}_0 z)$, we have

$$|d(f_t \circ h)(z)|_H/\eta_r(z) = |df_t(h(z))|_H/\eta_r(h(z))$$

by invariance of the Poincaré metric. Taking $g = f_t \circ h$ this gives the desired reparametrization.

3. **Proofs of Theorems 1 and 2.** Let V be a compact complex manifold, D a hypersurface with irreducible components $D_1, \ldots, D_m$, satisfying the hypotheses of Theorem 2. Choose a hermitian metric H on V. Cover V by open polydiscs $U_1, \ldots, U_N$, and choose an $\varepsilon > 0$ so that for any $p \in V$, the H-ball of radius ε in V lies inside one of these polydiscs. Further choose the U_i small enough that for all $p \in V$, the union of all U_i containing p is contained in a polydisc.

Let K_S denote, in general, the infinitesimal Kobayashi metric (see [8]) on a set S. The metric $K_{U_j - D_i}$ is complete, as it dominates K_{U_j} and hence is complete for ∂U_j, while as U_j is Stein, there is a holomorphic defining function α_{ji} for D_i in U_j so $U_j - D_i \to^{\alpha_{ij}} \Delta^*$, so $K_{U_j - D_i} \geqslant \alpha_{ij}^*(K_{\Delta^*})$, so $K_{U_j - D_i}$ is complete as we approach D_i. It is automatically upper-semicontinuous (see [8]), and lower-semicontinuity follows from completeness and the fact that

$U_j - D_i$ is contained in a polydisc by a standard normal families argument.

Let G_i be the differential metric on $V - D_i$ defined by

$$G_i = \min_{j=1,\ldots,N} K_{U_j - D_i},$$

where for each $p \in V - D_i$, the minimum is taken over those j so $p \in U_j$. The metric G_i is continuous as each $K_{U_j - D_i} \geqslant K_{U_j}$ and thus blows up on ∂U_j. It is complete, as at $p \in V$, G_i dominates K_{U-D_i}, where U is the polycylinder postulated in our choice of covering to contain the union of those U_i containing p. Take $G_0 = \max_{i=1,\ldots,m} G_i$ and finally set

$$G = \max(H, (\varepsilon/3)G_0).$$

Thus, if the infinitesimal Kobayashi pseudo-distance on $V - D$ dominates some nonzero multiple of G, we are done, as G is complete and $G \geqslant H$. If not, then there exists a $c > 0$ and a family of holomorphic maps

$$\Delta(r_i) \xrightarrow{f_i} V - D, \qquad r_i \to \infty,$$

with $|df_i(0)|_G > c$. By Brody's lemma, we can reparametrize to obtain a family $\Delta(r_i) \to^{g_i} V - D$ with $|dg_i(0)|_G = c$ and $|dg_i(z)|/\eta_{r_i}(z) \leqslant c$, all $z \in \Delta(r_i)$. Note $|dg_i(z)|_G \leqslant 4c/3$ if $z \in \Delta(r_i/2)$. The family g_i is equicontinuous (as $G \geqslant H$) and has a subsequence converging to a holomorphic map $C \to^g V$. Take this subsequence and reindex.

If $g(z) \not\in D$ for some $z \in \mathbf{C}$, then by the distance-decreasing property of the g_i and the completeness of the metric G, we have $\mathbf{C} \to^g V - D$. As $|dg(0)|_G = c$, the map is nonconstant, and as $|dg(z)|_G \leqslant c$ for all $z \in \mathbf{C}$ (as $\lim_{r_i \to \infty} \eta_{r_i}(z) = 1$), g is of finite order $\leqslant 2$. Otherwise, $\mathbf{C} \to^g D$. If D has several components $D_1, \ldots, D_m$, the preceding argument may be applied as $G \geqslant G_i$ and G_i is complete to show that either $g(\mathbf{C}) \cap D_i = \varnothing$ or $g(\mathbf{C}) \subset D_i$. So

$$\mathbf{C} \xrightarrow{g} D_{i_1} \cap \cdots \cap D_{i_k} - (D_{j_1} \cup \cdots \cup D_{j_l})$$

where $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{1, \ldots, m\}$.

The main thing is to see that g is nonconstant. By the distance-decreasing property of the g_i, we have for any $\rho \in (0, r_i/2)$ that $g_i(\Delta(\rho)) \subset B_{(4/3)(\rho c)}(g_i(0))$, where B_τ denotes, in general, the H-ball of radius τ. For i sufficiently large, we therefore have $g_i(\Delta(\rho)) \subset B_{(3/2)(\rho c)}(g(0))$.

If we take $\rho = 2\varepsilon/3c$, then $g_i(\Delta(\rho)) \subset B_\varepsilon(g(0))$, and hence lies in some U_j. So $\Delta(\rho) \to^{g_i} U_j - D$ and thus

$$|dg_i(0)|_{K_{U_j - D}} \leqslant 1/\rho = 3c/2\varepsilon,$$

so

$$(2\varepsilon/3)|dg_i(0)|_{K_{U_j - D}} \leqslant c = |dg_i(0)|_G.$$

Now

$$|dg_i(0)|_G = \max\big(|dg_i(0)|_H, (\varepsilon/3)|dg_i(0)|_{G_0}\big)$$

and

$$G_0 \leqslant \min_{j=1,\dots,N} K_{U_j-D}$$

as $K_{U_j-D_i} \leqslant K_{U_j-D}$ all $i = 1, \dots, m$. Hence

$$|dg_i(0)|_{G_0} \leqslant |dg_i(0)|_{K_{U_j-D}}$$

so

$$(\varepsilon/3)|dg_i(0)|_{G_0} \leqslant c/2 = (-1/2)|dg_i(0)|_G.$$

If $|dg_i(0)|G = (\varepsilon/3)|dg_i(0)|_{G_0}$ we would have a contradiction, so as $G = \max(H, (\varepsilon/3)G_0)$ we have $c = |dg_i(0)|_G = |dg_i(0)|_H$. Passing to the limit, $|dg(0)|_H = c$ and therefore g is nonconstant.

We thus obtain a nonconstant map g from $\mathbf{C}$ to either $V - D$ or a set of the form $D_{i_1} \cap \cdots \cap D_{i_k} - (D_{j_1} \cup \cdots \cup D_{j_l})$, $\{i_1, \dots, i_k, j_1, \dots, j_l\} = \{1, \dots, m\}$. Further, $|dg_i(z)|_H \leqslant c$, hence g is of finite order $\leqslant 2$. Under the hypotheses of Theorem 2, this cannot occur, and hence the infinitesimal Kobayashi metric on $V - D$ dominates a multiple of G, which proves the result.

If $V = \mathbf{P}_n$, $D = H_1 \cup \cdots \cup H_{2n+1}$, the H_i in general position, it follows that V and D satisfy the hypotheses of Theorem 2 by a classical result of E. Borel (see [5]) and the observation that $H_{i_1} \cap \cdots \cap H_{i_k} - (H_{j_1} \cup \cdots \cup H_{j_l})$, $\{i_1, \dots, i_k, j_1, \dots, j_l\} = \{1, 2, \dots, 2n+1\}$ is $\mathbf{P}_{n-k}$ minus $2n + 1 - k \geqslant 2(n-k) + 1$ hyperplanes in general position. We use only the Borel result for finite order maps, i.e., exponentials of polynomials, and it is easy in this case.

4. **Further results.** A consequence of Theorem 2 is

COROLLARY. *Let D be an algebraic curve of degree $\geqslant 5$ and genus $\geqslant 2$ in $\mathbf{P}_2$. Then $\mathbf{P}_2 - D$ is complete hyperbolic and hyperbolically embedded in $\mathbf{P}_2 \leftrightarrow$ there exists no nonconstant holomorphic map $\mathbf{C} \to \mathbf{P}_2 - D$ of finite order $\leqslant 2$.*

Thus, proving hyperbolicity is reduced to proving a Picard theorem. A slight modification of the argument gives the corollary for any D of degree $\geqslant 5$ with only ordinary double points.

A case where we can say more is $D = Q \cup L_1 \cup L_2 \cup L_3$ where Q is a nonsingular conic and L_1, L_2, L_3 are three lines in general position. We further assume the lines are not tangent to Q and no two of the lines intersect on Q, and that the tangents to Q at an intersection with one of the lines does not pass through the intersection of the other two lines. If $\mathbf{C} \xrightarrow{f} \mathbf{P} - D$, then lifting f to $\mathbf{C}^3$ and composing with the homogeneous polynomials defining L_1, L_2, L_3, Q and so labelled, we can arrange

$$L_i(f) = e^{\varphi_i}, \qquad i = 1, 2, 3, \qquad Q(f) = 1.$$

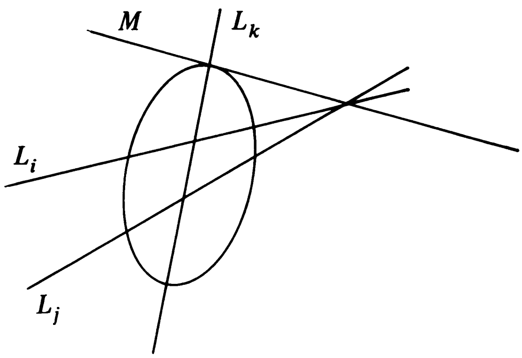
Now

$$Q = \sum_{i \leqslant j} A_{ij} L_i L_j \quad \text{so} \quad \sum_{i \leqslant j} A_{ij} e^{\varphi_i + \varphi_j} = 1.$$

Now by E. Borel's lemma on linear combinations of exponentials (see [5])

$e^{\varphi_i+\varphi_j}$ = constant, some i, j. Thus, $L_i(f)L_j(f) = cQ(f)$, some constant c.

Thus $f(\mathbf{C})$ lies in a conic or a line. But any conic meets $Q \cup L_1 \cup L_2 \cup L_3$ in at least 3 distinct points, hence f is constant by Picard's theorem. The only way a line M can meet $Q \cup L_1 \cup L_2 \cup L_3$ is less than 3 points is if we have the configuration



$\{i, j, k\} = \{1, 2, 3\}$

This was excluded, so again f must be constant. Thus

COROLLARY. *The complement of a generic configuration of a conic and three lines in $\mathbf{P}_2$ is complete hyperbolic and hyperbolically embedded in $\mathbf{P}_2$.*

By generic configuration is meant a Zariski open subset of the set of conics and three lines in $\mathbf{P}_2$. The forbidden configurations were listed earlier in this section.

REFERENCES

1. A. Bloch, *Sur les systèmes de fonctions holormorphes à variétés linéaires lacunaires*, Ann. Sci. École Norm Sup **43** (1926), 309–362.

2. R. Brody, *Intrinsic metrics and measures on compact complex manifolds*, Thesis, Harvard Univ., Cambridge, Mass.

3. H. Cartan, *Sur les systèmes de fonctions holomorphes à variétés lacunaires et leurs applications*, Ann. Sci. École Norm. Sup. **45** (1928), 255–346.

4. M. Cowen, *The Kobayashi metric on $\mathbf{P}_n - (2^n + 1)$ hyperplanes*, Value-Distribution Theory, Part A, Dekker, New York, 1974, pp. 205–224.

5. M. Green, *Some Picard theorems for holomorphic maps to algebraic varieties*, Amer. J. Math. **97** (1975), 43–75.

6. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Dekker, New York, 1970.

7. R. Nevanlinna, *Analytic functions*, Springer-Verlag, Berlin, 1970.

8. H. L. Royden, *Remarks on the Kobayashi metric*, Lecture Notes in Math., no. 185, Springer-Verlag, Berlin and New York, 1971, pp. 125–137.

9. P. Kiernan and S. Kobayashi, *Holomorphic mappings into projective space with lacunary hyperplanes*, Nagoya Math. J. **50** (1973), 199–216.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024