THE HYPERBOLICITY OF THE COMPLEMENT OF $2n + 1$
HYPERPLANES IN GENERAL POSITION IN $\mathbb{P}^n$, AND
RELATED RESULTS

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Abstract. Using a modified version of a technique of R. Brody, a simple
proof is found that the complement of $2n + 1$ hyperplanes in general
position in $\mathbb{P}^n$ is complete hyperbolic and hyperbolically embedded in $\mathbb{P}^n$. In
fact, a more general result is obtained showing that a suitable Picard
theorem is sufficient to imply hyperbolicity in a large class of algebro-geo-
metric situations.

1. Introduction. Our main theorem, first proved by A. Bloch [1] (see also [3])
in 1926, is the following:

Theorem 1. The complement of $2n + 1$ hyperplanes in general position in $\mathbb{P}^n$
is complete hyperbolic and hyperbolically embedded in $\mathbb{P}^n$.

(For the meaning of these terms, see [6] and [9].)

Bloch's proof is exceedingly complex, and there has been considerable
interest in finding a simpler argument. A differential-geometric argument of
M. Cowen [4] gives a nice proof for $n = 2$, but breaks down for higher $n$. The
method of proof here will be a noncompact version of a simple but fertile
technique introduced by R. Brody [2]. A similar proof has been found
independently by A. Howard.

We will obtain the more general result:

Theorem 2. Let $D$ be a union of (possibly singular) hypersurfaces
$D_1, \ldots, D_m$ in a compact complex manifold $V$. Then $V - D$ is complete
hyperbolic and hyperbolically embedded in $V$ provided

1. there are no nonconstant holomorphic maps $C \to V - D$, and
2. there are no nonconstant holomorphic maps $C \to D_{i_1} \cap \cdots \cap D_{i_k} - (D_{j_1}
\cup \cdots \cup D_{j_l})$, for any choice of distinct indices so $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} =
\{1, \ldots, m\}$.

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In fact, we need only assume there are no nonconstant holomorphic maps of finite order \(< 2\) in (1) and (2).

(For the meaning of finite order, see [7].)
Some applications of Theorem 2 other than \(\mathbb{P}^n\) minus hyperplanes will be discussed in §4.

2. Brody's reparametrization lemma. Let \(M\) be a complex manifold, possibly with boundary. Let \(H\) be a continuous differential metric, i.e. a continuous function \(|\varphi|_H\) on the tangent bundle of \(M\) satisfying \(|\alpha \varphi_x| = |\alpha| \varphi_x|_H\) for all \(\alpha \in \mathbb{C}, \varphi_x \in T_x(M)\). Let \(\Delta(r)\) be the disc of radius \(r\) and
\[
\eta_r(z) = \frac{r^2}{(r^2 - |z|^2)}.
\]

**Brody's Reparametrization Lemma.** Let \(\Delta(r) \to^f M\) be holomorphic, with \(|df(0)|_H > c\). Then there exists \(\Delta(r) \to^g M\) holomorphic so
\[
\begin{align*}
(1) \quad &|g(0)| = c, \\
(2) \quad &|dg(z)|_H/\eta_r(z) < c, \text{ all } z \in \Delta(r), \\
(3) \quad &\text{Image}(g) \subset \text{Image}(f).
\end{align*}
\]

**Proof.** Let \(\Delta(r) \to^f M\) be given by \(f_t(z) = f(tz)\). Then
\[
\frac{|df_t(z)|_H}{\eta_r(z)} = t \frac{r^2 - |z|^2}{r^2 - |tz|^2} \cdot \frac{|df_k(tz)|_H}{\eta_r(tz)}.
\]

If \(u(t) = \sup_{z \in \Delta(r)} |df_t(z)|_H/\eta_r(z)\), then:
\[
(1) \quad u(t) \text{ is finite, monotone increasing, and continuous on } [0, 1),
(2) \quad c < u(1) < \infty.\text{Thus, for suitable } t \in [0, 1), \text{ we have } u(t) = c. \text{ The maximum occurs at a point } z_0 \in \Delta(r), \text{ and if } h(z) = (z - z_0)/(1 - \bar{z}z_0), \text{ we have}
\]
\[
|d(f_t \circ h)(z)|_H/\eta_r(z) = |df_t(h(z))|_H/\eta_r(h(z))
\]
by invariance of the Poincaré metric. Taking \(g = f_t \circ h\) this gives the desired reparametrization.

3. Proofs of Theorems 1 and 2. Let \(V\) be a compact complex manifold, \(D\) a hypersurface with irreducible components \(D_1, \ldots, D_m\), satisfying the hypotheses of Theorem 2. Choose a hermitian metric \(H\) on \(V\). Cover \(V\) by open polydiscs \(U_1, \ldots, U_N\), and choose an \(\epsilon > 0\) so that for any \(p \in V\), the \(H\)-ball of radius \(\epsilon\) in \(V\) lies inside one of these polydiscs. Further choose the \(U_i\) small enough that for all \(p \in V\), the union of all \(U_i\) containing \(p\) is contained in a polydisc.

Let \(K_S\) denote, in general, the infinitesimal Kobayashi metric (see [8]) on a set \(S\). The metric \(K_{U_j - D_j}\) is complete, as it dominates \(K_{U_j}\) and hence is complete for \(\partial U_j\), while as \(U_j\) is Stein, there is a holomorphic defining function \(a_j\) for \(D_j\) in \(U_j\) so \(U_j - D_j \to a_j(K_{\Delta^*})\), so \(K_{U_j - D_j} \supset \alpha_j(K_{\Delta^*})\), so \(K_{U_j - D_j}\) is complete as we approach \(D_j\). It is automatically upper-semicontinuous (see [8]), and lower-semicontinuity follows from completeness and the fact that
Let $G_i$ be the differential metric on $V - D_i$ defined by

$$G_i = \min_{j=1, \ldots, N} K_{U_j - D_i},$$

where for each $p \in V - D_i$, the minimum is taken over those $j$ so $p \in U_j$. The metric $G_i$ is continuous as each $K_{U_j - D_i} \supset K_{U_j}$ and thus blows up on $\partial U_j$. It is complete, as at $p \in V$, $G_i$ dominates $K_{U_j - D_i}$, where $U$ is the polycylinder postulated in our choice of covering to contain the union of those $U_i$ containing $p$. Take $G_0 = \max_{i=1, \ldots, m} G_i$ and finally set

$$G = \max(G, (\varepsilon/3)G_0).$$

Thus, if the infinitesimal Kobayashi pseudo-distance on $V - D$ dominates some nonzero multiple of $G$, we are done, as $G$ is complete and $G \geq H$. If not, then there exists a $c > 0$ and a family of holomorphic maps

$$A(r_i) \to V - D, \quad r_i \to \infty,$$

with $|df_i(0)| \geq c$. By Brody's lemma, we can reparametrize to obtain a family $A(r_i) \to V - D$ with $|dg_i(0)| = c$ and $|dg_i(z)|/n_i(z) < c$, all $z \in \Delta(r_i)$. Note $|dg_i(z)| \leq 4c/3$ if $z \in \Delta(r_i/2)$. The family $g_i$ is equicontinuous (as $G \geq H$) and has a subsequence converging to a holomorphic map $C \to V$. Take this subsequence and reindex.

If $g(z) \not\in D$ for some $z \in C$, then by the distance-decreasing property of the $g_i$ and the completeness of the metric $G$, we have $C \to V - D$. As $|dg(0)|_G = c$, the map is nonconstant, and as $|dg_i(z)|_G \leq c$ for all $z \in C$ (as $\lim_{r_i \to \infty} n_i(\tau_i) = 1$), $g$ is of finite order $< 2$. Otherwise, $C \to D$. If $D$ has several components $D_1, \ldots, D_m$, the preceding argument may be applied as $G \geq G_i$ and $G_i$ is complete to show that either $g(C) \cap D_i = \emptyset$ or $g(C) \subset D_i$. So

$$C \to D_{i_1} \cap \cdots \cap D_{i_k} \setminus (D_{j_1} \cup \cdots \cup D_{j_l})$$

where $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{1, \ldots, m\}$.

The main thing is to see that $g$ is nonconstant. By the distance-decreasing property of the $g_i$, we have for any $\rho \in (0, r_i/2)$ that $g_i(\Delta(\rho)) \subset B_{(4/3)(\rho c)}(g_i(0))$, where $B_\tau$ denotes, in general, the $H$-ball of radius $\tau$. For $i$ sufficiently large, we therefore have $g_i(\Delta(\rho)) \subset B_{(3/2)(\rho c)}(g(0))$.

If we take $\rho = 2\varepsilon/3c$, then $g_i(\Delta(\rho)) \subset B_{c}(g(0))$, and hence lies in some $U_j$. So $\Delta(\rho) \to U_j - D$ and thus

$$|dg_i(0)|_{K_{U_j - D}} \leq 1/\rho = 3c/2\varepsilon,$$

so

$$(2\varepsilon/3)|dg_i(0)|_{K_{U_j - D}} \leq c = |dg_i(0)|_G.$$

Now

$$|dg_i(0)|_G = \max(|dg_i(0)|_H, (\varepsilon/3)|dg_i(0)|_{G_0})$$

and
\[ G_0 < \min_{j=1, \ldots, N} K_{U_j - D} \]
as \( K_{U_j - D} \leq K_{U_j - D} \) all \( i = 1, \ldots, m \). Hence
\[ |d g_i(0)|_{g_0} < |d g_i(0)|_{K_{U_j - D}} \]
so
\[ (\varepsilon/3)|d g_i(0)|_{g_0} < c/2 = (-1/2)^{\mu} \]
If \( |d g_i(0)|_{G} = (\varepsilon/3)|d g_i(0)|_{g_0} \) we would have a contradiction, so as \( G = \max(H, (\varepsilon/3) G_0) \) we have \( c = |d g_i(0)|_{G} = |d g_i(0)|_{H} \). Passing to the limit, \( |d g(0)|_{H} = c \) and therefore \( g \) is nonconstant.
We thus obtain a nonconstant map \( g \) from \( C \) to either \( V - D \) or a set of the form \( D_{i_1} \cap \cdots \cap D_{i_k} \) if \( D_{i_1} \cup \cdots \cup D_{i_k} \), \( \{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{1, \ldots, m\} \). Further, \( |d g(z)|_{H} < c \), hence \( g \) is of finite order < 2. Under the hypotheses of Theorem 2, this cannot occur, and hence the infinitesimal Kobayashi metric on \( V - D \) dominates a multiple of \( G \), which proves the result.
If \( V = \mathbb{P}^n, D = H_1 \cup \cdots \cup H_{2n+1} \), the \( H_i \) in general position, it follows that \( V \) and \( D \) satisfy the hypotheses of Theorem 2 by a classical result of E. Borel (see [5]) and the observation that \( H_{i_1} \cap \cdots \cap H_{i_k} \cup (H_{j_1} \cup \cdots \cup H_{j_l}), \{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{1, 2, \ldots, 2n+1\} \) is \( \mathbb{P}^n - k \) minus \( 2n + 1 - k \) > 2(n - k) + 1 hyperplanes in general position. We use only the Borel result for finite order maps, i.e., exponentials of polynomials, and it is easy in this case.

4. Further results. A consequence of Theorem 2 is

**Corollary.** Let \( D \) be an algebraic curve of degree > 5 and genus > 2 in \( \mathbb{P}^2 \). Then \( \mathbb{P}^2 - D \) is complete hyperbolic and hyperbolically embedded in \( \mathbb{P}^2 \leftrightarrow \) there exists no nonconstant holomorphic map \( C \rightarrow \mathbb{P}^2 - D \) of finite order < 2.

Thus, proving hyperbolicity is reduced to proving a Picard theorem. A slight modification of the argument gives the corollary for any \( D \) of degree > 5 with only ordinary double points.
A case where we can say more is \( D = Q \cup L_1 \cup L_2 \cup L_3 \) where \( Q \) is a nonsingular conic and \( L_1, L_2, L_3 \) are three lines in general position. We further assume the lines are not tangent to \( Q \) and no two of the lines intersect on \( Q \), and that the tangents to \( Q \) at an intersection with one of the lines does not pass through the intersection of the other two lines. If \( C \rightarrow \mathbb{P} - D \), then lifting \( f \) to \( \mathbb{C}^3 \) and composing with the homogeneous polynomials defining \( L_1, L_2, L_3, Q \) and so labelled, we can arrange
\[ L_i(f) = e^{\xi_i}, \quad i = 1, 2, 3, \quad Q(f) = 1. \]
Now
\[ Q = \sum_{i<j} A_{ij} L_i L_j \quad \text{so} \quad \sum_{i<j} A_{ij} e^{\xi_i + \xi_j} = 1. \]
Now by E. Borel's lemma on linear combinations of exponentials (see [5])
$e^{\pi i} + \theta = \text{constant, some } i, j$. Thus, $L_i(f)L_j(f) = cQ(f)$, some constant $c$.

Thus $f(C)$ lies in a conic or a line. But any conic meets $Q \cup L_1 \cup L_2 \cup L_3$ in at least 3 distinct points, hence $f$ is constant by Picard’s theorem. The only way a line $M$ can meet $Q \cup L_1 \cup L_2 \cup L_3$ is less than 3 points is if we have the configuration

\[\{i, j, k\} = \{1, 2, 3\}\]

This was excluded, so again $f$ must be constant. Thus

**Corollary.** The complement of a generic configuration of a conic and three lines in $\mathbb{P}_2$ is complete hyperbolic and hyperbolically embedded in $\mathbb{P}_2$.

By generic configuration is meant a Zariski open subset of the set of conics and three lines in $\mathbb{P}_2$. The forbidden configurations were listed earlier in this section.

**References**