PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 66, Number 1, September 1977

# ON THE EXISTENCE OF $\omega$-POINTS

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ABSTRACT. Without additional axioms to ZFC it is shown that each point of a dense-in-itself Hausdorff space in which nonempty $G_\delta$'s have nonempty interiors is an $\omega$-point.

For a cardinal number $\gamma$ and a point $p$ of a topological space $X$, we say that $p$ is a $\gamma$-point of $X$ if $p$ lies simultaneously on the boundaries of (at least) $\gamma$ pairwise disjoint open subsets of $X$. N. Hindman [5] has shown that if the Continuum Hypothesis ($\omega^+ = 2^\omega$) is assumed, then every point of $\beta\omega - \omega$, the space of nonprincipal ultrafilters on the countably infinite discrete space $\omega$, is a $2^\omega$-point. The same conclusion has been established by E. van Douwen [4] from the hypothesis $P(2^\omega)$: If $\mathfrak{F}$ is a family of subsets of $\omega$, the cardinality of $\mathfrak{F}$ ($|\mathfrak{F}|$) is less than $2^\omega$ and $\bigcap \mathfrak{T}$ is infinite for each finite subfamily $\mathfrak{T} \subset \mathfrak{F}$, then there is an infinite $S \subset \omega$ such that $S - F$ is finite for each $F \in \mathfrak{F}$. The hypothesis $P(2^\omega)$ follows from Martin's Axiom [1], which is a consequence of the Continuum Hypothesis, and is essentially weaker than Martin's Axiom [6]. As a generalization of Hindman's result to higher cardinals can be seen a result by K. Prikry [8], who proved that if $\alpha$ is regular and $\alpha^+ = 2^\alpha$, then every point of $U(\alpha)$, the space of all uniform ultrafilters on $\alpha$, is a $2^\alpha$-point of $U(\alpha)$.

In the proofs of all known results concerning the problem of whether each point of $U(\alpha)$ is a $\gamma$-point of $U(\alpha)$ for a $\gamma \geqslant 2$ and an $\alpha \geqslant \omega$, some additional set-theoretic assumptions were used. This naturally leads to the question of whether that problem can be solved without any special set-theoretic assumptions. Some results in this direction were recently obtained by W. W. Comfort and N. Hindman [2]. Here we show that each nonisolated point of a Hausdorff space $X$, in which nonempty $G_\delta$'s have nonempty interiors, is an $\omega$-point of $X$ whenever the cellularity of $X$ or the character of that point is not Ulam-measurable. This implies that each point of $\beta\omega - \omega$ is an $\omega$-point of $\beta\omega - \omega$, which is an answer to a question raised by W. W. Comfort (a letter), E. van Douwen [4] and R. Telgàrsky (oral communication).

I would like to express my thanks to the referee and to Eric van Douwen. Their comments on this paper were of great help to the author.

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Received by the editors December 30, 1975 and, in revised form, August 20, 1976 and March 29, 1977.

*AMS (MOS) subject classifications* (1970). Primary 54G99, 54A25.

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**Results.** Before, the following definitions are needed.

A *$P^*$-point* of a space $X$ is a point which lies in the interior of every intersection of countable many regularly open sets containing the point.

A family $\mathcal{U}$ of pairwise disjoint open subsets of a space $X$ is a *cellular family for* $p$, $p \in X$, if $p \notin \operatorname{cl} U$ for $U \in \mathcal{U}$ and $p \in \operatorname{cl} \bigcup \mathcal{U}$. If $p$ is a nonisolated point of a Hausdorff space $X$, then there exists a cellular family for $p$ of cardinality not greater than $\chi(p, X)$, the character of $p$, and there exists a cellular family for $p$ of cardinality not greater than $c(X)$, the cellularity of $X$.

A cardinal $\alpha$ is *Ulam-measurable* if there is a nonprincipal ultrafilter $\xi$ on $\alpha$ such that $\bigcap \mathcal{F} \in \xi$ whenever $\mathcal{F} \subset \xi$ and $|\mathcal{F}| \leqslant \omega$. Let us recall some known facts (see [7]) which will be used in the sequel without explicit mention.

(1) $\omega$ is not Ulam-measurable;

(2) if $\alpha$ is not Ulam-measurable and $\beta \leqslant \alpha$, then $\beta$, as well as $2^\alpha$, are not Ulam-measurable.

LEMMA 1. *Let $p$ be a $P^*$-point of a space $X$ and let $\mathcal{U}$ be a cellular family for $p$. If $|\mathcal{U}|$ is not Ulam-measurable, then there exists $\mathcal{U}' \subset \mathcal{U}$ such that $p \in \operatorname{cl} \bigcup \mathcal{U}' \cap \operatorname{cl} \bigcup (\mathcal{U} - \mathcal{U}')$.*

PROOF. Consider the subfamilies $\mathcal{A}$ of $\mathcal{U}$ such that $p \notin \operatorname{cl} \bigcup A$, and then consider the set $\xi$ consisting of all subfamilies $\mathcal{U} - \mathcal{A}$ of $\mathcal{U}$ for $\mathcal{A}$ as above. If $F \subset \xi$ and $F$ is countable, then $\bigcap F \in \xi$. Since $\mathcal{U} - \{U\} \in \xi$ for each $U \in \mathcal{U}$ and $|\mathcal{U}|$ is not Ulam-measurable, $\xi$ is not an ultrafilter on $\mathcal{U}$. Therefore, there exists $\mathcal{U}' \subset \mathcal{U}$ such that $\mathcal{U}' \notin \xi$ and $\mathcal{U} - \mathcal{U}' \notin \xi$. There is $p \in \operatorname{cl} \bigcup \mathcal{U}' \cap \operatorname{cl} \bigcup (\mathcal{U} - \mathcal{U}')$.

LEMMA 2. *Let $p$ be a $P^*$-point of a space $X$ and let $\mathcal{U}$ be a cellular family for $p$. If $|\mathcal{U}|$ is not Ulam-measurable, then $p$ is an $\omega$-point of $X$.*

PROOF. If $\mathcal{T} \subset \mathcal{U}$, then $|\mathcal{T}|$ is not Ulam-measurable. Using Lemma 1, we construct inductively a sequence $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \dots$ of subfamilies of $\mathcal{U}$ such that $p \in \operatorname{cl} \bigcup \mathcal{U}_n \cap \operatorname{cl} \bigcup (\mathcal{U}_{n-1} - \mathcal{U}_n)$ for each $n$. Since the members of $\mathcal{U}$ are pairwise disjoint, the sets $V_n = \bigcup (\mathcal{U}_{n-1} - \mathcal{U}_n)$ are pairwise disjoint and $p \in \operatorname{cl} V_n$ for each $n$. Thus $p$ is an $\omega$-point.

THEOREM. *Let $p$ be a nonisolated point of a Hausdorff space $X$ in which nonempty $G_\delta$'s have nonempty interior. If $c(X)$ or $\chi(p, X)$ is not Ulam-measurable, then $p$ is an $\omega$-point of $X$.*

PROOF. *Let* $\mathcal{M}$ be a maximal, possibly empty, family of pairwise disjoint open subsets of $X$ such that

(1) $p \in \operatorname{cl} U$ for each $U \in \mathcal{M}$;

(2) each $U \in \mathcal{M}$ is contained in an $F_\delta$-subset of $X$ which does not contain $p$.

Then $p$ clearly is an $\omega$-point of $X$ if $\mathcal{M}$ is infinite.

If $\mathcal{M}$ is finite (or even countable), then it easily follows from (1) and (2) that $p$ is an accumulation point of the set $I = \operatorname{Int} \bigcap \{X - U: U \in \mathfrak{M}\}$, since nonempty $G_\delta$'s have nonempty interiors. Let $Y$ be the subspace $\{p\} \cup I$ of $X$. Clearly, $c(Y) = c(I) \leqslant c(X)$, and $\chi(p, X) \geqslant \chi(p, Y)$. Hence there exists a cellular family for $p$ in $Y$ having not Ulam-measurable cardinality, since $c(X)$ or $\chi(p, X)$ is not Ulam-measurable. So if we can show that $p$ is a $P^*$-point of $Y$, then it follows from Lemma 2 that $p$ is an $\omega$-point of $Y$, hence of $X$, since $Y - \{p\} = I$ is open in $X$.

It remains to show that $p$ is a $P^*$-point of $Y$. Suppose not. Then there is a family $\{V_n: n < \omega\}$ of regularly open subsets of $Y$, each containing $p$, such that $p \notin \operatorname{Int}_Y \bigcap \{V_n: n < \omega\}$, but then also

$$p \notin \operatorname{Int}_Y \bigcap \{\operatorname{cl}_Y V_n: n < \omega\} = \operatorname{Int}_Y\big(Y \cap \bigcap \{\operatorname{cl} V_n: n < \omega\}\big).$$

Hence the set

$$W = Y - \bigcap \{\operatorname{cl} V_n: n < \omega\} = I - \bigcap \{\operatorname{cl} V_n: n < \omega\}$$

is an open subset of $X$ with $p \in \operatorname{cl} W$. As $W \subset I$ and $W \subset \bigcup \{X - V_n: n < \omega\}$, this contradicts the maximality of $\mathfrak{M}$. This completes the proof.

REMARK. It might be worthwhile to point out, according to a comment by the referee, that in the main theorem the condition that $c(X)$ or $\chi(p, X)$ be not Ulam-measurable is superfluous if $p$ is not a $P^*$-point.

COROLLARY. *If $\alpha$ is not Ulam-measurable, then each point of $\beta(\alpha) - \alpha$ is an $\omega$-point of $\beta(\alpha) - \alpha$.*

PROOF. The space $\beta(\alpha) - \alpha$ is a dense-in-itself (compact) Hausdorff space in which nonempty $G_\delta$'s have nonempty interiors and $c(\beta(\alpha) - \alpha) \leqslant 2^\alpha$ [3].

## REFERENCES


1. D. Booth, *Ultrafilters on a countable set*, Ann. Math. Logic **2** (1970), 1–24.
2. W. W. Comfort and N. Hindman, *Refining families for ultrafilters*, Math. Z. **149** (1976), 189–199.
3. W. W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Grundlehren Math. Wiss., Bd. 211, Springer-Verlag, Berlin, 1974.
4. E. van Douwen, *Martin's axiom and pathological points in $\beta X - X$* (preprint).
5. N. Hindman, *On the existence of c-points in $\beta N - N$*, Proc. Amer. Math. Soc. **21** (1969), 227–280.
6. K. Kunen and F. D. Tall, *Between Martin's axiom and Souslin's hypothesis* (preprint).
7. K. Kuratowski and A. Mostowski, *Set theory*, North-Holland, Amsterdam; PWN, Warsaw, 1967.
8. K. Prikry, *Ultrafilters and almost disjoint sets*, Bull. Amer. Math. Soc. **81** (1975), 209–212.


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