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SCHUR MULTIPLIERS OF SOME FINITE NILPOTENT GROUPS

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ABSTRACT. Let B denote the Burnside group, $B(p^\alpha, d)$ and let $G = B/B_k$ where p is a prime and $1 < k < p$. We show that the Schur multiplier, $M(G)$, is a direct power of $\Psi(k, d)$ cyclic groups, each having order p^α, where $\Psi(k, d) = k^{-1}\sum_{n|k}\mu(k/n)d^n$. (This is Witt's formula for the rank of F_k/F_{k+1} where F is free on d generators.) In addition we can show that $M(B(3, d))$ is elementary abelian of exponent 3 and rank $2\binom{d}{2} + 4\binom{d}{3} + 3\binom{d}{4}$.

1. **Notation and a preliminary result.** For any group G, $M(G)$ will be the Schur multiplier of G. The minimal number of generators for G will be denoted $d(G)$. When G is abelian we will sometimes refer to $d(G)$ as the rank of G and write $\mathrm{rk}(G)$. $Z(G)$ will be the center of G and G' will be the commutator subgroup of G. G_k will be the kth term of the lower central series $G_1 = G, G_2 = G' = (G, G), G_3 = (G_2, G), \ldots$.

Throughout this paper d will be a fixed positive integer, p will be a fixed prime and q will be p^α for some fixed positive integer α. F will be a free group on a set $\{x_1, x_2, \ldots, x_d\}$ of d generators, and $\Psi(k, d)$, the rank of F_k/F_{k+1}, will be abbreviated $\Psi(k)$. $B(q, d)$ will be the Burnside group with exponent q on d generators.

I thank N. Blackburn for showing me a proof of the following generalization of a theorem of R. C. Lyndon [7].

THEOREM 1.1. *Let $B = B(q, d)$. Then B_k/B_{k+1} is a direct sum of $\Psi(k)$ cyclic groups of order q, for $k < p$.*

PROOF. We claim that $F_k \cap F^qF_{k+1} = F_k^qF_{k+1}$ for $k < p$, hence

$$B_k/B_{k+1} \cong F^qF_k/F^qF_{k+1} \cong F_k/F_k \cap F^qF_{k+1} \cong F_k/F_k^qF_{k+1}.$$

To prove the claim we show by induction on j, for $1 \leqslant j \leqslant k$, that $F_j \cap F^qF_{k+1} \subseteq F_j^qF_{k+1}$. Given $w \in F_j \cap F^qF_{k+1}$, then $w \in F_{j-1} \cap F^qF_{k+1}$, so by induction $w \in F_{j-1}^qF_{k+1}$. After moving the factors from F_{k+1} to the right, we can write $w = a_1^q \cdots a_r^qx$ where $a_i \in F_{j-1}$ and $x \in F_{k+1}$. Now consider $(a_1a_2 \cdots a_r)^q$ and apply Theorem 12.3.1 of [2] to get

$$(a_1a_2 \cdots a_r)^q = a_1^qa_2^q \cdots a_r^qc_1^{e_1} \cdots c_t^{e_t}y$$

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where $y \in F_{k+1}$, each $c_i \in F_j$ and q divides each e_i since $k < p$. Observe that $wx^{-1} = a_1^q \cdots a_r^q \in F_j$, so $(a_1 a_2 \cdots a_r)^q \in F_j$. Since F/F_j is torsion free, $a_1 a_2 \cdots a_r \in F_j$. But then since q divides each l_i, we get $a_1^q \cdots a_r^q \in F_j^q F_{k+1}$.

2. The exponent of $M(G)$.

THEOREM 2.1. *Let $B = B(q, d)$, $G = B/B_k$ where $k < 2p$ and $Q = G/N$ where $N \subseteq G_l$ and $l > k - p$. Then $M(Q)$ has exponent dividing q.*

PROOF. We may assume that d is chosen so that $d(Q) = d(G) = d$ and that k is chosen so that class Q = class $G = k - 1$.

Consider an exact sequence $M \mapsto H \twoheadrightarrow^{\pi} Q$ where $M \subseteq H' \cap Z(H)$. Then H is a finite p-group with $d(H) = d$ and class $H \leqslant k$. Let $\{z_1, z_2, \ldots, z_d\}$ be a set of generators for Q and let $\{h_1, h_2, \ldots, h_d\}$ be a set of generators for H with $\pi h_i = z_i$. For any commutator $\bar{c}$ in Q with entries from $\{z_i\}$ we will let c denote the corresponding commutator in H with each z_i replaced by h_i.

Observe that for any h_i, $h_i^q \in M$, and thus is central. Hence given $c = (h_{i_1}, \ldots, h_{i_j})$, we have $(cH_{j+1})^q = (h_{i_1}, \ldots, h_{i_j}^q)H_{j+1} = H_{j+1}$. It follows that H_j/H_{j+1} has exponent dividing q. We wish to show that H_{k-p+1} has exponent dividing q. We show by induction on λ, for $0 \leqslant \lambda < p$, that $H_{k-\lambda}$ has exponent dividing q. Given $\lambda < p$, write $r = k - \lambda$, so that $r + p > k$. Then for any commutator $c = (h_{i_1}, \ldots, h_{i_r}) \in H_r$ we use, e.g., Lemma H2 of [9] to write

$$1 = (h_{i_1}, \ldots, h_{i_r}^q) = c^q c_1^{e_1} \cdots c_t^{e_t}$$

where the c_i are all commutators with $r < wtc_i \leqslant k$. Since $p + r > k$, q divides each e_i. By induction the c_i have exponent dividing q, so c has exponent dividing q as well. For an arbitrary element x of $H_{k-\lambda}$, we may write x as a product of commutators of weight at least $k - \lambda$ and apply Theorem 12.3.1 of [2] together with the above result for commutators to show that x has exponent dividing q.

Thus to prove the theorem it suffices to show that $M \subseteq H_{k-p+1}$. Since $l \geqslant k - p + 1$ and $p \geqslant k - p + 1$ we are done if we show either that $M \subseteq H_l$ or that $M \subseteq H_p$. We find the remainder of the argument easier to follow if we separate the cases $l < p$ and $l \geqslant p$.

Case (i) $l < p$. We show in this case that $M \subseteq H_l$. For each j, choose a basis $\{\bar{c}_1 Q_{j+1}, \ldots, \bar{c}_{r(j)} Q_{j+1}\}$ for Q_j/Q_{j+1}, taking $\{z_1 Q_2, z_2 Q_2, \ldots, z_d Q_2\}$ as the choice for Q/Q_2. Fix a weight preserving order of all of the $\bar{c}$ and relabel according to this order, so that (up to the choice of bases and the choice of the order) each element w of Q has a unique expression $w = \prod \bar{c}_i^{\bar{\alpha}_i}$ where $\bar{c}_i \in Q_j$, $\bar{c}_i Q_{j+1}$ has order $\bar{\beta}_i$ in Q_j/Q_{j+1} and $0 \leqslant \bar{\alpha}_i < \bar{\beta}_i$.

For $j < l$, $Q_j/Q_{j+1} \cong B_j/B_{j+1}$, hence $r(j) = \Psi(j)$ and Q_j/Q_{j+1} is homocyclic of exponent q. We have seen above that H_j/H_{j+1} has exponent dividing q. Since Q_j/Q_{j+1} is an image of H_j/H_{j+1}, the latter must have rank $\Psi(j)$ and exponent q for $j < l$. The corresponding sets

$\{c_1H_{j+1}, \ldots, c_{\Psi(j)}H_{j+1}\}$ form bases for H_j/H_{j+1} when $j < l$. Choose, as well, bases for H_j/H_{j+1} when $j \geqslant l$. Order those c_i with weight less than l in the same way as the $\bar{c}_i$ were ordered, and extend this order to a weight preserving order of all of the c_i's. Then each element h of H has a unique expression $h = \prod c_i^{\alpha_i}$ where $c_i \in H_j$, c_iH_{j+1} has order β_i in H_j/H_{j+1} and $0 \leqslant \alpha_i < \beta_i$. If we apply the quotient map π to h, we get

$$\pi h = \pi \prod c_i^{\alpha_i} = \prod \pi c_i^{\alpha_i} = \prod_{wtc_i < l} \bar{c}_i^{\alpha_i} \cdot \prod_{wtc_i \geqslant l} \bar{c}_i^{\alpha_i'}$$

so that the unique expression for πh has $\bar{\alpha}_i = \alpha_i$ for $wtc_i < l$ and $\bar{\alpha}_i$ equal to some α_i' when $wtc_i \geqslant l$. If $h \in M$, then $\pi h = 1$, so $\bar{\alpha}_i = 0$ for all i. Hence $\alpha_i = 0$ for $wtc_i < l$ and $h \in H_l$.

Case (ii) $l \geqslant p$. The argument here is the same as in Case (i) except we observe that $Q_j/Q_{j+1} \cong B_j/B_{j+1} \cong H_j/H_{j+1}$ for $j < p$ and show that $M \subseteq H_p$.

COROLLARY 2.2. *Let $B = B(q, d)$, $G = B/B_k$ where $p \leqslant k < 2p$ and let $M \rightarrowtail H \twoheadrightarrow G$ be exact with $M \subseteq H' \cap Z(H)$. Then $M \subseteq H_p$.*

COROLLARY 2.3. *Let $B = B(q, d)$ and $G = B/B_k$ where $k < 2p$. Then $M(G)$ has exponent dividing q.*

COROLLARY 2.4. *If Q is any group of exponent* 3, *then $M(Q)$ has exponent dividing* 3.

PROOF. Let $d = d(Q)$ and $B = B(3, d)$. Then $B = B/B_4$ (see [2]) and $Q = B/N$ for some $N \subseteq B_2$.

Corollary 2.4 is implicit in Remark 2.8 of Jones [5] since any group of exponent 3 satisfies the 2nd Engel condition.

We do not have any example to show that the condition on k in Theorem 2.1 is necessary. The example of Bayes, Kautsky, and Wamsley [1] of a group G with exponent 4 with $M(G)$ having exponent 8 shows that the hypothesis on l cannot be dropped.

3. **The multiplier of groups having exponent** q.

THEOREM 3.1. *If $B = B(q, d)$ and $G = B/B_k$ where $1 < k < p$, then $M(G)$ is homocyclic with exponent q and rank $\Psi(k)$.*

PROOF. Observe first that $B_k/B_{k+1} \rightarrowtail B/B_{k+1} \twoheadrightarrow G$ is exact with B_k/B_{k+1} in both the commutator and central subgroups of B/B_{k+1}. It follows from a theorem of Schur, see [3, V, Satz 23.5], that B_k/B_{k+1} is an epimorphic image of $M(G)$. Then by Theorem 1.1 we see that rk $M(G) \geqslant \Psi(k)$ and $|M(G)| \geqslant q^{\Psi(k)}$.

By Corollary 2.3, $M(G)$ has exponent dividing q, so it will suffice to show that rk $M(G) \leqslant \Psi(k)$. To this end we exhibit a presentation for G.

Let $\{c_1F_{k+1}, \ldots, c_{\Psi(k)}F_{k+1}\}$ be a basis for F_k/F_{k+1}. Then we claim that G has a presentation

$$\langle x_1, x_2, \ldots, x_d | x_1^q, x_2^q, \ldots, x_d^q, c_1, \ldots, c_{\Psi(k)}, F_{k+1} \rangle.$$

Let $\bar{G}$ be the group having this presentation. The obvious function $\bar{G} \to G$ preserves relations, and hence is a group epimorphism. $\bar{G}$ is nilpotent of class $k-1$ and $\operatorname{rk} \bar{G}_j/\bar{G}_{j+1} \leqslant \Psi(j) = \operatorname{rk} G_j/G_{j+1}$. We argue as in Theorem 2.1 that each $\bar{G}_j/\bar{G}_{j+1}$ has exponent dividing q. It follows that $|\bar{G}_j/\bar{G}_{j+1}| \leqslant |G_j/G_{j+1}|$ so that $|\bar{G}| \leqslant |G|$.

Having this presentation for G, it follows, as in [3, V, Satz 25.2], that $\operatorname{rk} M(G) \leqslant \Psi(k)$. Since G is nilpotent of class $k-1$, the relators in F_{k+1} contribute nothing to the number of generators of $M(G)$.

It would be of interest to know if F_{k+1} can be omitted from the set of relators in the above presentation. If so, the resulting presentation has deficiency $-\operatorname{rk} M(G)$.

PROPOSITION 3.2. *If $B = B(q, d)$ and $G = B/B_p$ then $M(G)$ has exponent dividing q and* $\operatorname{rk} M(G) \leqslant \Psi(p)$.

PROOF. Let $M \rightarrowtail H \twoheadrightarrow G$ be exact with $M \subseteq H' \cap Z(H)$. By Corollary 2.2, $M \subset H_p$. But $H_{p+1} = 1$, so $\operatorname{rk} H_p \leqslant \Psi(p)$.

Using results of Lyndon [7], one can easily show that

$$\operatorname{rk} M(G) \geqslant \Psi(p) + d - \binom{p+d-1}{p} \quad \text{for } G = B/B_p.$$

The difficulty in improving the result here is that we do not have a convenient representation group to exhibit.

4. **The multipliers of groups of exponent 3.** Throughout this section we let $B = B(3, d)$. B has been described by Levi and van der Waerden [6]. For details see, for example, [2]. Our present proofs of results in this section are tedious, but straightforward, and we condense or omit them.

LEMMA 4.1. *B has presentations with d generators and $r = d + 2\binom{d}{2} + 4\binom{d}{3} + 3\binom{d}{4}$ relators.*

PROOF. One such presentation is

$$\begin{aligned}
\langle x_1, x_2, \ldots, x_d | x_i^3 &= 1, 1 \leqslant i \leqslant d, \\
(x_j, x_i, x_i) &= (x_j, x_i, x_j) = 1, 1 \leqslant i < j \leqslant d, \\
(x_k, x_j, x_i, x_i) &= (x_k, x_j, x_i, x_j) = (x_k, x_j, x_i, x_k) \\
&= (x_k, x_j, x_i)(x_k, x_i, x_j) = 1, 1 \leqslant i < j < k \leqslant d, \\
(x_l, x_k, x_j, x_i) &= (x_l, x_k, x_i, x_j) = (x_l, x_j, x_i, x_k) = 1, \\
&\qquad 1 \leqslant i < j < k < l \leqslant d \rangle.
\end{aligned}$$

LEMMA 4.2. *There is a group H and a short exact sequence $M \rightarrowtail H \twoheadrightarrow B$ where $M \subseteq Z(H) \cap H'$ and M is elementary abelian of exponent 3 and rank $r - d = 2\binom{d}{2} + 4\binom{d}{3} + 3\binom{d}{4}$.*

The presentation in Lemma 4.1 can be regarded as a blueprint for

constructing such a representation group H. The details are tedious to verify.

THEOREM 4.3. *$M(B)$ is elementary abelian of exponent* 3 *and rank* $r - d = 2\binom{d}{2} + 4\binom{d}{3} + 3\binom{d}{4}$.

PROOF. This follows easily from Lemmas 4.1 and 4.2.

We remark that MacDonald [8] has previously computed the rank of $M(B(3, 3))$ as an application of a computer algorithm.

For the case $q = 3$, we can improve on Proposition 3.2.

PROPOSITION 4.2. *Let $G = B/B_3$. Then $M(G)$ has exponent* 3 *and rank* $\Psi(3)$.

Representation groups for $G = B/B_3$ are much easier to construct than those for B. We note, however, that, with a proper choice H of a representation group for B, H/H_4 is a representation group for G.

As a final remark, we note that essentially all of the multipliers that we are obtaining are nontrivial and that this is not surprising in view of D. L. Johnson's recent paper [4].

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