SUCCESSIVE DERIVATIVES OF ENTIRE FUNCTIONS

SIMON HELLERSTEIN\(^1\) AND JACK WILLIAMSON\(^2\)

Abstract. We show that if \( f \) is a real entire function which has, along with each of its derivatives, only real nonpositive zeros, then either \( f(z) = ce^{az}, c \) and \( a \) real constants, or

\[
f(z) = cz^m e^{az} \prod_{n} \left(1 + \frac{z}{|z_n|} \right)
\]

where \( a > 0 \) and \( \sum |z_n|^{-1} < \infty \).

This note concerns a subclass of the class \( \mathcal{U}_0 \) of entire functions \( g \) of the form

\[
g(z) = cz^m e^{-\gamma z^2 + az} \prod_{n} \left(1 - \frac{z}{z_n} \right)e^{z/z_n}
\]

where \( c \) is a constant, \( \gamma > 0, a \) and the \( z_n \) are real, and \( \sum |z_n|^{-2} < \infty \). The subclass of \( \mathcal{U}_0 \) that we will be interested in is the class \( \mathcal{U} \) of entire functions \( f \) of the form

\[
f(z) = cz^m e^{az} \prod_{n} \left(1 + \frac{z}{|z_n|} \right)
\]

where \( a > 0 \) and \( \sum |z_n|^{-1} < \infty \).

The class \( \mathcal{U}_0 \) (often called the Laguerre-Pólya class) and its subclass \( \mathcal{U} \) are of special interest since classical theorems of Laguerre [3] and Pólya [4] assert that \( f \in \mathcal{U}_0(\mathcal{U}) \) if and only if \( f \) can be uniformly approximated on discs about the origin by a sequence of polynomials with only real (real nonpositive) zeros. A corollary of their results is: if \( f \in \mathcal{U}_0(\mathcal{U}) \) if and only if \( f \) can be uniformly approximated on discs about the origin by a sequence of polynomials with only real (real nonpositive) zeros \( n = 1, 2, \ldots \); in particular, \( f \in \mathcal{U}_0(\mathcal{U}) \) implies \( f^{(n)} \) has only real (real nonpositive) zeros \( n = 1, 2, \ldots \). In 1915, Pólya [6] (see also [5]) asked whether the following converse assertion holds: If a (constant multiple of a) real entire function \( f \) (i.e., \( z \) real implies \( f(z) \) real) and each of its derivatives \( f^{(n)}, n = 1, 2, \ldots \) have only real (real nonpositive) zeros is \( f \in \mathcal{U}_0(\mathcal{U}) \)?

Pólya showed [5], [6] that if \( f(z) = P(z)e^{Q(z)} \) where \( P \) and \( Q \) are polynomials, then the answer to this question is affirmative. Recently the authors

\[\text{(Received by the editors December 13, 1976 and, in revised form, March 21, 1977.)} \quad \text{AMS (MOS) subject classifications (1970). Primary 30A66.} \]

\[\text{Key words and phrases. Real entire function, derivative, genus of a canonical product.} \]

\[\text{\(^1\)Supported in part by NSF grant MPS 75-07000.} \]

\[\text{\(^2\)Supported in part by NSF grant MPS 75-07098.} \]

\[\text{© American Mathematical Society 1977}\]
established [1], [2] the full conjecture for $\mathcal{U}_0$ with the following

**Theorem A.** Let $f$ be a (constant multiple of a) real entire function. If $f, f', \text{ and } f''$ have only real zeros, then $f \in \mathcal{U}_0$.

In this note we use this result to establish the conjecture for $\mathcal{U}$ with the

**Theorem B.** Let $f$ be a (constant multiple of a) real entire function. If $f$ and $f^{(n)}$ have only real nonpositive zeros $n = 1, 2, \ldots$, then either $f(z) = ce^{az},$ or $f \in \mathcal{U}$.

Before proving Theorem B, we remark that its hypotheses cannot be weakened by requiring only that $f$ and $f^{(n)}, n = 1, 2, \ldots, N$, for some $N$ have only real nonpositive zeros – see Remark 2 below.

**Proof of Theorem B.** From Theorem A we see that

\[
(3) \quad f(z) = cz^m e^{-\gamma z^2 + \sigma z} \prod \Pi(z)
\]

where $c$ is a constant which, without loss of generality, we take to be 1, and

where $\Pi(z)$ is a canonical product of genus $< 1$ with only negative zeros. Thus, it only remains to show that if $f(z) \neq ce^{az}$, then (i) $\gamma = 0$, (ii) $\Pi$ is of genus 0, and (iii) $\sigma > 0$. To do this we first observe that there is no loss in generality in assuming that $f$ and $f^{(k)}, \quad k = 1, 2, \ldots$, have only negative zeros. Indeed, if $f^{(k)}$ has a zero at the origin for some $k$ ($f^{(0)} = f$), consider $g(z) = f(z + \epsilon)$ where $\epsilon > 0$. Then $g$ and $g^{(k)}, \quad k = 1, 2, \ldots$, have only negative zeros, and if this implies $g \in \mathcal{U}$, then clearly $f \in \mathcal{U}$. Proceeding under this assumption then, we set $m = 0$ in (3) and next observe that since $f \in \mathcal{U}_0, f(z) \neq ce^{az}$, either $f$ or $f^{(k)}, \quad k = 1, 2, \ldots$, must have some (negative) zeros. Also, if

\[
(4) \quad f(z) = 1 + \sum_{n=1} \frac{c_n z^n}{n!}
\]

then

\[
(5) \quad c_n = f^{(n)}(0) \neq 0, \quad n = 1, 2, \ldots;
\]

moreover, it cannot be the case that $c_n c_{n+1} < 0$ for $n = 0, 1, 2, \ldots$ for then

\[
(6) \quad (-1)^k f^{(k)}(-x) = \sum_{n=0} (-1)^n c_{n+k} \frac{x^n}{n!} > 0 \quad \text{for } x = \Re z > 0
\]

and $f^{(k)}$ cannot have any negative zeros $k = 0, 1, 2, \ldots (f^{(0)} = f)$. Thus, there is at least one nonnegative integer $n$ for which $c_n c_{n+1} > 0$; choose such an $n$ and denote it by $k$, so that

\[
(7) \quad f^{(k)}(z) = c_k e^{-\gamma_k z^2 + \sigma_k z} \prod_k \Pi_k(z)
\]

where $\gamma_k > 0, \quad \sigma_k$ is a real constant, and $\Pi_k$ is a canonical product, finite or infinite, of genus $< 1$ with only negative zeros. Taking the logarithmic
We now need the following elementary facts about the growth of canonical products of finite genus with only negative zeros: If $P(z)$ is a canonical product of genus $q$ with only negative zeros $\{-a_n\}$, $a_n > 0$, then

$$
\frac{P'(z)}{P(z)} = (-1)^q q \sum_{n} \frac{1}{a_n^q (z + a_n)}, \quad z \neq -a_n,
$$

from which it follows readily that for $x = \Re z$

$$
\left| \frac{P'(x)}{P(x)} \right| = o(x^q) \quad (x \to +\infty)
$$

and

$$
\lim_{x \to +\infty} \frac{P'(x)}{P(x)} = -\infty \quad (q \text{ odd}).
$$

We will now establish (i)–(iii). Since, as is easily verified, $\gamma_k = \gamma$ and genus $\Pi_k = \text{genus } \Pi$,

$$
(8) \quad \lim_{x \to +\infty} \frac{f^{(k+1)}(x)}{f^{(k)}(x)} = -\infty \quad \gamma > 0 \quad (x = \Re z).
$$

Further, (5) and (6) imply that

$$
(9) \quad f(z) = e^{a\pi i \Pi}(z), \quad f^{(k)}(z) = c_k e^{a_k \pi i \Pi_k}(z),
$$

where, as is easily verified, $\sigma = \sigma_k$. It then follows from (8) with $\beta = 0$ and (9) that $\sigma = \sigma_k < 0$, then

$$
\lim_{x \to +\infty} \frac{f^{(k+1)}(x)}{f^{(k)}(x)} = \sigma_k < 0 \quad (x = \Re z).
$$

Reasoning as above, (13) yields a contradiction. Thus $\sigma > 0$, and the proof of Theorem B is complete.

Remark 1. If we alter the hypothesis of Theorem B by dropping the condition that $f$ be real but requiring instead that $f$ be of finite order, we can then use Theorem 2 of [1] and the above proof to conclude that either $f(z) = ae^{bz}$, $a$ and $b$ constants, or $f \in \mathbb{H}$.

3If $f \in \mathbb{H}_0$, then $f'$ has only real zeros and $(f'/f)(x) < 0$. Thus $f'$ has exactly one zero, a simple one, between two consecutive zeros of $f$, so that the genus of $\Pi_1 = \text{genus of } \Pi$. A simple growth argument shows that $\gamma_1 = \gamma$. Introduction completes the argument. In the case $\gamma = 0$, cf. E. Borel, *Leçons sur les fonctions entières*, Paris, 1921, p. 32.
Remark 2. Theorem B is false if it is only required that $f$ and $f^{(n)}$, $n = 1, 2, \ldots, N$, for some $N$ have only real nonpositive zeros. To see this, let $f$ be of the form (3) with only real nonpositive zeros, $f \in \mathcal{U}$. By Laguerre's inequality, $\left| \frac{f^{(n)}(z)}{f^{(n-1)}(z)} \right|' < 0$ for $z \neq$ zero of $f^{(n-1)}$; thus, if we denote by $z_{1N}$ the largest zero of $f^{(N)}$, it is clear that all zeros of $f$ and $f^{(n)}$ lie in $(-\infty, z_{1N})$, $n = 1, 2, \ldots, N$. If $z_{1N} > 0$, let $g(z) = f(z + z_{1N})$ and if $z_{1N} < 0$, let $g(z) = f(z)$. In either case $g$ and $g^{(n)}$, $n = 1, 2, \ldots, N$, have only real nonpositive zeros; however, $g \notin \mathcal{U}$.

Added in proof. Part of our proof involved showing that if $f \in \mathcal{U}_0$ and $f^{(k)}$ has only real nonpositive zeros for all $k$, then $\gamma = 0$ in (1). This also follows from a theorem of Edrei [Scripta Math. 22 (1956), Theorem 1] which readily implies the following

**Theorem C.** Let $F(z) = e^{-\gamma z^2}G(z)$, $\gamma > 0$ and $G \in \mathcal{U}_0$ of genus $< 1$. Then the zeros of the successive derivatives of $f$ are everywhere dense on the real axis.

**References**


Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use