SUCCESSIVE DERIVATIVES OF ENTIRE FUNCTIONS

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ABSTRACT. We show that if \( f \) is a real entire function which has, along with each of its derivatives, only real nonpositive zeros, then either \( f(z) = ce^{\sigma z}, c \) and \( \sigma \) real constants, or

\[
f(z) = cz^m e^{\alpha z} \prod_n \left(1 + \frac{z}{|z_n|}\right)
\]

where \( \sigma > 0 \) and \( \sum_n |z_n|^{-1} < \infty \).

This note concerns a subclass of the class \( \mathcal{U}_0 \) of entire functions \( g \) of the form

\[
g(z) = cz^m e^{-\gamma z^2 + \alpha z} \prod_n \left(1 - \frac{z}{z_n}\right)e^{z/z_n}
\]

where \( c \) is a constant, \( \gamma > 0, \sigma \) and the \( z_n \) are real, and \( \sum_n |z_n|^{-2} < \infty \). The subclass of \( \mathcal{U}_0 \) that we will be interested in is the class \( \mathcal{U} \) of entire functions \( f \) of the form

\[
f(z) = cz^m e^{\alpha z} \prod_n \left(1 + \frac{z}{|z_n|}\right)
\]

where \( \sigma > 0 \) and \( \sum_n |z_n|^{-1} < \infty \).

The class \( \mathcal{U}_0 \) (often called the Laguerre-Pólya class) and its subclass \( \mathcal{U} \) are of special interest since classical theorems of Laguerre [3] and Pólya [4] assert that \( f \in \mathcal{U}_0(\mathcal{U}) \) if and only if \( f \) can be uniformly approximated on discs about the origin by a sequence of polynomials with only real (real nonpositive) zeros. A corollary of their results is: \( f \in \mathcal{U}_0(\mathcal{U}) \) if and only if \( f \) can be uniformly approximated on discs about the origin by a sequence of polynomials with only real (real nonpositive) zeros. A corollary of their results is: \( f \in \mathcal{U}_0(\mathcal{U}) \) if and only if \( f \) can be uniformly approximated on discs about the origin by a sequence of polynomials with only real (real nonpositive) zeros. A corollary of their results is: \( f \in \mathcal{U}_0(\mathcal{U}) \) if and only if \( f \) can be uniformly approximated on discs about the origin by a sequence of polynomials with only real (real nonpositive) zeros. A corollary of their results is: \( f \in \mathcal{U}_0(\mathcal{U}) \) if and only if \( f \) can be uniformly approximated on discs about the origin by a sequence of polynomials with only real (real nonpositive) zeros.

In 1915, Pólya [6] (see also [5]) asked whether the following converse assertion holds: If a (constant multiple of a) real entire function \( f \) (i.e., \( z \) real implies \( f(z) \) real) and each of its derivatives \( f^{(n)} \), \( n = 1, 2, \ldots \) have only real (real nonpositive) zeros is \( f \in \mathcal{U}_0(\mathcal{U}) \)?

Pólya showed [5], [6] that if \( f(z) = P(z)e^{Q(z)} \) where \( P \) and \( Q \) are polynomials, then the answer to this question is affirmative. Recently the authors

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established [1], [2] the full conjecture for \( \mathcal{U}_0 \) with the following

**Theorem A.** Let \( f \) be a (constant multiple of a) real entire function. If \( f, f', \) and \( f'' \) have only real zeros, then \( f \in \mathcal{U}_0 \).

In this note we use this result to establish the conjecture for \( \mathcal{U} \) with the

**Theorem B.** Let \( f \) be a (constant multiple of a) real entire function. If \( f \) and \( f^{(n)} \) have only real nonpositive zeros \( n = 1, 2, \ldots \), then either \( f(z) = ce^{az} \), \( a \) real, or \( f \in \mathcal{U}_0 \).

Before proving Theorem B, we remark that its hypotheses cannot be weakened by requiring only that \( f \) and \( f^{(n)} \), \( n = 1, 2, \ldots, N \), for some \( N \) have only real nonpositive zeros – see Remark 2 below.

**Proof of Theorem B.** From Theorem A we see that

\[
(3) \quad f(z) = cz^m e^{-\gamma z^2 + \sigma z} \prod (z)
\]

where \( c \) is a constant which, without loss of generality, we take to be 1, and where \( \prod(z) \) is a canonical product of genus \( < 1 \) with only negative zeros. Thus, it only remains to show that if \( f(z) \neq ce^{az} \), then (i) \( \gamma = 0 \), (ii) \( \prod \) is of genus \( 0 \), and (iii) \( \sigma > 0 \). To do this we first observe that there is no loss in generality in assuming that \( f \) and \( f^{(k)} \), \( k = 1, 2, \ldots \), have only negative zeros. Indeed, if \( f^{(k)} \) has a zero at the origin for some \( k \) \( (f^{(0)} = f) \), consider \( g(z) = f(z + \epsilon) \) where \( \epsilon > 0 \). Then \( g \) and \( g^{(k)} \), \( k = 1, 2, \ldots \), have only negative zeros, and if this implies \( g \in \mathcal{U}_0 \), then clearly \( f \in \mathcal{U}_0 \). Proceeding under this assumption then, we set \( m = 0 \) in (3) and next observe that since \( f \in \mathcal{U}_0, f(z) \neq ce^{az} \), either \( f \) or \( f^{(k)}, k = 1, 2, \ldots \), must have some (negative) zeros. Also, if

\[
(4) \quad f(z) = 1 + \sum_{n=1}^{\infty} c_n z^n
\]

then

\[
(5) \quad c_n = f^{(n)}(0) \neq 0, \quad n = 1, 2, \ldots ;
\]

moreover, it cannot be the case that \( c_n c_{n+1} < 0 \) for \( n = 0, 1, 2, \ldots \) for then

\[
(6) \quad (-1)^k f^{(k)}(-x) = \sum_{n=0}^{\infty} (-1)^{n+k} c_{n+k} \frac{x^n}{n!} > 0 \quad \text{for } x = \Re z > 0
\]

and \( f^{(k)} \) cannot have any negative zeros \( k = 0, 1, 2, \ldots \) \( (f^{(0)} = f) \). Thus, there is at least one nonnegative integer \( n \) for which \( c_n c_{n+1} > 0 \); choose such an \( n \) and denote it by \( k \), so that

\[
(6) \quad c_k c_{k+1} > 0.
\]

Since, as was pointed out above, \( f \in \mathcal{U}_0 \) implies \( f^{(k)} \in \mathcal{U}_0 \), we write

\[
(7) \quad f^{(k)}(z) = c_k e^{-\gamma_k z^2 + \sigma_k z} \prod_k(z)
\]

where \( \gamma_k > 0 \), \( \sigma_k \) is a real constant, and \( \prod_k \) is a canonical product, finite or infinite, of genus \( < 1 \) with only negative zeros. Taking the logarithmic
derivative of (7), we have
\begin{equation}
\frac{f^{(k+1)}}{f^{(k)}}(z) = -2\gamma_k z + \sigma_k + \frac{\Pi'_k(z)}{\Pi_k(z)}.
\end{equation}

We now need the following elementary facts about the growth of canonical products of finite genus with only negative zeros: If \( P(z) \) is a canonical product of genus \( q \) with only negative zeros \( \{-a_n\}, a_n > 0 \), then
\begin{equation}
\frac{P'(z)}{P(z)} = (-1)^q q \sum_n \frac{1}{a_n^q (z + a_n)}, \quad z \neq -a_n,
\end{equation}
from which it follows readily that for \( x = \Re z \)
\begin{equation}
\left| \frac{P'}{P}(x) \right| = o(x^q) \quad (x \to +\infty)
\end{equation}
and
\begin{equation}
\lim_{x \to +\infty} \frac{P'}{P}(x) = -\infty \quad (q \text{ odd}).
\end{equation}

We will now establish (i)-(iii). Since, as is easily verified, \( \gamma_k = \gamma \) and genus \( \Pi_k = \text{genus } \Pi,^3 \) (8)-(10) imply that if either \( \gamma > 0 \) or \( \Pi \) is of genus 1 then
\begin{equation}
\lim_{x \to +\infty} \frac{f^{(k+1)}}{f^{(k)}}(x) = -\infty \quad (x = \Re z).
\end{equation}
Further, (5) and (6) imply that
\begin{equation}
\frac{f^{(k+1)}}{f^{(k)}}(0) > 0.
\end{equation}
It now follows from (11) and (12) that \( f^{(k+1)} \) has a positive zero, which is a contradiction. Thus \( \gamma = \gamma_k = 0 \), \( \Pi \) and \( \Pi_k \) are of genus 0, and, by (3) (with \( m = 0 \)) and (7)
\begin{equation}
f(z) = e^{\alpha z} \Pi(z), \quad f^{(k)}(z) = c_k e^{\alpha_k z} \Pi_k(z),
\end{equation}
where, as is easily verified, \( \sigma = \sigma_k \). It then follows from (8) (with \( \gamma_k = 0 \)) and (9) (with \( q = 0 \)) that if \( \sigma = \sigma_k < 0 \), then
\begin{equation}
\lim_{x \to +\infty} \frac{f^{(k+1)}}{f^{(k)}}(x) = \sigma_k < 0 \quad (x = \Re z).
\end{equation}
Reasoning as above, (13) yields a contradiction. Thus \( \sigma > 0 \), and the proof of Theorem B is complete.

Remark 1. If we alter the hypothesis of Theorem B by dropping the condition that \( f \) be real but requiring instead that \( f \) be of finite order, we can then use Theorem 2 of [1] and the above proof to conclude that either \( f(z) = ae^{bz}, \) \( a \) and \( b \) constants, or \( f \in \mathcal{Q} \).

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3If \( f \in \mathcal{Q}_0 \), then \( f' \) has only real zeros and \( (f'/f)(x) < 0 \). Thus \( f' \) has exactly one zero, a simple one, between two consecutive zeros of \( f \), so that the genus of \( \pi_1 = \text{genus of } \pi \). A simple growth argument shows that \( \gamma_1 = \gamma \). Introduction completes the argument. In the case \( \gamma = 0 \), cf. E. Borel, *Leçons sur les fonctions entières*, Paris, 1921, p. 32.
Remark 2. Theorem B is false if it is only required that $f$ and $f^{(n)}$, $n = 1, 2, \ldots, N$, for some $N$ have only real nonpositive zeros. To see this, let $f$ be of the form (3) with only real nonpositive zeros, $f \notin \mathcal{U}$. By Laguerre's inequality, $[f^{(n)}(z)/f^{(n-1)}(z)]' < 0$ for $z \neq$ zero of $f^{(n-1)}$; thus, if we denote by $z_{1N}$ the largest zero of $f^{(N)}$, it is clear that all zeros of $f$ and $f^{(n)}$ lie in $(-\infty, z_{1N})$, $n = 1, 2, \ldots, N$. If $z_{1N} > 0$, let $g(z) = f(z + z_{1N})$ and if $z_{1N} < 0$, let $g(z) = f(z)$. In either case $g$ and $g^{(n)}$, $n = 1, 2, \ldots, N$, have only real nonpositive zeros; however, $g \notin \mathcal{U}$.

Added in proof. Part of our proof involved showing that if $f \in \mathcal{U}_0$ and $f^{(k)}$ has only real nonpositive zeros for all $k$, then $\gamma = 0$ in (1). This also follows from a theorem of Edrei [Scripta Math. 22 (1956), Theorem 1] which readily implies the following

**Theorem C.** Let $F(z) = e^{-\gamma z^2}G(z)$, $\gamma > 0$ and $G \in \mathcal{U}_0$ of genus $< 1$. Then the zeros of the successive derivatives of $f$ are everywhere dense on the real axis.

**References**