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UNITARY PARTS OF CONTRACTIVE HANKEL MATRICES

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ABSTRACT. For a Hankel matrix $H = (c_{j+k})$ which is a contraction, necessary and sufficient conditions are obtained for the existence of a nontrivial unitary part, and an explicit description of this unitary part is given.

1. **Introduction**. In what follows, l^2 will denote the usual class of square-summable complex sequences, L^2 will denote the class of Lebesgue measurable functions on the unit circle in the complex plane, and H^2 will denote the Hardy (closed) subspace of L^2 consisting of those functions in L^2 whose Fourier coefficients vanish on the negative integers. Functions which differ only on zero sets will be considered equal.

With a sequence $(c_0, c_1, \ldots)$ in l^2 we associate the *Hankel matrix* $H = (c_{j+k})$, where $j, k = 0, 1, 2, \ldots,$ acting on l^2. If we define $\phi(e^{i\theta}) = c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \ldots$, then such a matrix may be realized as an operator on H^2, defined by

$$Hx(e^{i\theta}) = P_+\phi(e^{i\theta})x(e^{-i\theta}),$$

where $P_+ \colon L^2 \to H^2$ is the orthogonal projection.

A result of Nehari [4] states that a Hankel matrix H is bounded if and only if there exists a function $f \in L^\infty$ such that

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta, \tag{1.1}$$

and, in this case, f can be chosen so that $\|f\|_\infty = \|H\|$. Such a function f is called a *minifunction* of H.

We shall consider Hankel matrices which are contractions ($\|Hx\| \leqslant \|x\|$) on the Hardy space H^2. If a Hankel matrix is a contraction, then, by the theorem of Nehari, we can find a minifunction $f \in L^\infty$ such that $\|f\|_\infty = \|H\| \leqslant 1$ and $Hx(e^{i\theta}) = P_+ f(e^{i\theta})x(e^{-i\theta})$ for $x \in H^2$.

2. **The unitary part of a Hankel contraction.** Following Sz.-Nagy and Foiaş [5], we say that a contraction T on a Hilbert space K is *completely nonunitary* if T has no nontrivial reducing subspace N such that the restriction $T|N$ of T to N is unitary. It is known [5, Theorem I.3.2] that for any contraction T on K we can find a unique orthogonal decomposition $K = M \oplus M_1$ such that $T|M$ is unitary and $T|M_1$ is completely nonunitary. It is not excluded that M or

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M_1 is possibly the subspace $\{0\}$. Furthermore, M is given by $M = \{x \in K: \|T^n x\| = \|x\| = \|T^{*n} x\|, n = 1, 2 \dots \}$ and is called the *unitary subspace* of T. $T|M$ is called the *unitary part* of T.

If $\psi(e^{i\theta}) \in L^\infty$, then the corresponding bounded *Toeplitz operator* T_ψ: $H^2 \to H^2$ is defined by T_ψ: $x(e^{i\theta}) \mapsto P_+ \psi(e^{i\theta}) x(e^{i\theta})$ for $x \in H^2$. In [1] Goor proved that if T_ψ is a Toeplitz contraction ($\|T_\psi\| \leqslant 1$, i.e., $|\psi(e^{i\theta})| \leqslant 1$ a.e.), then T_ψ is completely nonunitary unless ψ is a constant. This result may be used to obtain necessary and sufficient conditions for the existence of a nontrivial unitary part of a Hankel contraction. These conditions, together with a characterization of the unitary part, when it exists, are obtained as a corollary to the following.

THEOREM 2.1. *Let $H = (c_{j+k})$ be a Hankel contraction. Then H will have a nontrivial unitary subspace M only when there exists a minifunction $f(e^{i\theta})$ for H such that*

$$|f(e^{i\theta})| = 1 \text{ a.e. (so that } \|H\| = 1\text{), and} \tag{2.1}$$

$$f(e^{i\theta}) f(e^{-i\theta}) = k^2 \text{ a.e. for some constant } k^2, \ |k| = 1. \tag{2.2}$$

In such a case, the minifunction $f(e^{i\theta})$ is unique, and M is given by the three equivalent expressions:

$$M = \left[e^{i\theta} f(e^{i\theta}) H^2\right]^\perp \cap H^2, \quad \text{where the orthogonal complement is in } L^2, \tag{2.3}$$

$$M = \{x \in H^2: H^* H x = x\}, \tag{2.4}$$

$$M = \{x \in H^2: \bar{k} H x = x\} \oplus \{x \in H^2: \bar{k} H x = -x\}. \tag{2.5}$$

PROOF. Suppose that a Hankel contraction H has a nontrivial unitary subspace. Then there exists some $x \neq 0$ in H^2 such that for H and its adjoint $H^* = \bar{H}$ we have $\|H^n x\| = \|x\| = \|\bar{H}^n x\|$ for $n = 1, 2, \dots$. Taking $n = 1$ gives that

$$\|x\| = \|Hx\| = \|P_+ f(e^{i\theta}) x(e^{-i\theta})\| \leqslant \|f(e^{i\theta}) x(e^{-i\theta})\| \leqslant \|x(e^{-i\theta})\|.$$

This implies that $P_+ f(e^{i\theta}) x(e^{-i\theta}) = f(e^{i\theta}) x(e^{-i\theta})$, so that $Hx = f(e^{i\theta}) x(e^{-i\theta}) \in H^2$. It also immediately follows that the minifunction $f(e^{i\theta})$ must therefore be unique, cf. [2, p. 863]. Furthermore, we may apply a well-known corollary of the F. and M. Riesz theorem [3, p. 52] to the equality

$$\|f(e^{i\theta}) x(e^{-i\theta})\| = \|x(e^{-i\theta})\|$$

to conclude that $|f(e^{i\theta})| = 1$ almost everywhere on the unit circle. This establishes (2.1).

Taking now $n = 2$, we get $\|x\| = \|H^2 x\| = \|Hf(e^{i\theta}) x(e^{-i\theta})\| = \|P_+ f(e^{i\theta}) f(e^{-i\theta}) x(e^{i\theta})\| \leqslant \|f(e^{i\theta}) f(e^{-i\theta}) x(e^{i\theta})\| \leqslant \|x(e^{i\theta})\|$, so that

$$H^2 x = f(e^{i\theta}) f(e^{-i\theta}) x(e^{i\theta}) = T_\psi x,$$

where $\psi(e^{i\theta}) = f(e^{i\theta}) f(e^{-i\theta})$. Continuing, we obtain

(2.6) $\quad H^{2n+1}x(e^{i\theta}) = f^{n+1}(e^{i\theta})f^n(e^{-i\theta})x(e^{-i\theta}), \qquad n = 0, 1, \ldots,$

(2.7) $\quad H^{2n}x(e^{i\theta}) = \left[f(e^{i\theta})f(e^{-i\theta})\right]^n x(e^{i\theta}), \qquad n = 1, 2, \ldots.$

But (2.7) together with its analogue for $\bar{H}^{2n}$ (obtained by replacing $f(e^{i\theta})$ by $\bar{f}(e^{i\theta})$) implies that the Toeplitz operator T_ψ has a nontrivial unitary part. Therefore, by Goor's result, $f(e^{i\theta})f(e^{-i\theta}) = k^2$ for some constant k, $|k| = 1$. Hence, $\bar{k}f(e^{i\theta}) = k\bar{f}(e^{-i\theta})$, so that, by (1.1), $\bar{k}c_n$ is real for all n.

Formulas (2.6) and (2.7) now reduce to

$$H^{2n+1}x(e^{i\theta}) = k^{2n}Hx(e^{i\theta}), \qquad n = 0, 1, 2, \ldots,$$
$$H^{2n}x(e^{i\theta}) = k^{2n}x(e^{i\theta}), \qquad n = 1, 2, \ldots,$$

valid for all x in the unitary subspace of H. Similar expressions are easily obtained for $\bar{H}^{2n+1}x(e^{i\theta})$ and $\bar{H}^{2n}x(e^{i\theta})$.

The maximal subspace on which H is unitary now becomes $M = \{x \in H^2\colon \|Hx\| = \|x\|\} = \{x \in H^2\colon H^*Hx = x\}$, giving (2.4). Furthermore, we then have $x \in M \Leftrightarrow f(e^{i\theta})x(e^{-i\theta}) \in H^2 \Leftrightarrow f(e^{-i\theta})x(e^{i\theta}) \perp e^{i\theta}H^2 \Leftrightarrow x(e^{i\theta}) \perp e^{i\theta}\bar{f}(e^{-i\theta})H^2 \Leftrightarrow x \in [e^{i\theta}f(e^{i\theta})H^2]^\perp \cap H^2$, since $f(e^{-i\theta}) = k^2\bar{f}(e^{i\theta})$. This establishes (2.3). Finally, since the matrix $\bar{k}H$ is real, hence selfadjoint, with $\|\bar{k}H\| = 1$, we get (2.5). This completes the proof.

As a result of the above theorem, we then obtain a characterization of those Hankel matrices having nontrivial unitary subspaces.

COROLLARY 2.1. *Let H be a Hankel contraction. Then a necessary and sufficient condition that H have a nontrivial unitary subspace is that there exist a constant k, $|k| = 1$, such that $\bar{k}H$ is real (hence selfadjoint) and that M in (2.5) satisfy $M \neq \{0\}$.*

Using a result of Sz.-Nagy and Foiaş for completely nonunitary contractions, we can define a functional calculus for functions $u \in H^\infty$. In particular, if $B(H^2)$ is the space of bounded operators on H^2, we then have the following.

COROLLARY 2.2. *Let H be a Hankel contraction. If either $\|H\| < 1$ of kH is not real for all $k \in \mathbf{C}$, then the map $u \to u(H)$ from the Hardy space H^∞ into $B(H^2)$ defined by*

$$u(H) = \text{strong} \lim_{r \to 1-} \sum_{k=0}^{\infty} a_k r^k H^k,$$

where $u(e^{i\theta}) = \sum_{k=0}^\infty a_k e^{ik\theta}$, is a contractive homomorphism of the algebra H^∞ into $B(H^2)$.

This follows from the above theorem and Theorem III.2.1 of [5].

Theorem 2.1 can also be used to establish a property of the point spectrum of a bounded Hankel matrix.

COROLLARY 2.3. *Let H be a bounded Hankel matrix. If $\lambda = \|H\|$ is an eigenvalue of H, then H is real. Hence, if αH is not real for any nonzero* *(complex constant)* α, *then* $|\lambda| < \|H\|$ *holds for all eigenvalues* λ *of* H *(if any)*.

PROOF. Without loss of generality, we can suppose that $\|H\| = 1$. (In such a case, the associated eigenspace $M = \{y \in H^2\colon Hy = y\}$ is well-known to be a reducing subspace for the operator H [5, Proposition I.3.1].) If $y \neq 0$ satisfies $Hy = y$, then, by the proof of Theorem 2.1, $\overline{H} = \bar{\alpha}H$ and $H^2y = \alpha y$ for some $|\alpha| = 1$. But since $H^2y = y$, we get $\alpha = 1$, and hence $\overline{H} = \bar{\alpha}H = H$, which shows that H must be real.

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