THE CLASS OF COMPACT* SPACES IS PRODUCTIVE AND CLOSED HEREDITARY

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Abstract. W. W. Comfort defined compact* spaces as completely regular Hausdorff spaces $X$ such that each maximal ideal in the ring $C^*(X, \mathbb{R})$ of bounded continuous real-valued functions on $X$ is fixed. He showed that, independently of the axiom of choice, the class of compact* spaces is productive and closed hereditary. We give a short new proof of this.

If $X, E$ are topological spaces and $L \subseteq C(X, E)$, then a mapping $H: L \to E$ will be said to have an obvious representation iff there is a point $x_0 \in X$ such that $H(f) = f(x_0)$ for all $f \in L$.

In his Theorem 2.4* W. W. Comfort [1] proved that $C^*(X, \mathbb{R})/M$ is just the real field $\mathbb{R}$ whenever $M$ is a maximal ideal of $C^*(X, \mathbb{R})$. So his Definition 3.1* is equivalent to the following

Definition 1. A space $X$ is compact* iff it is a completely regular Hausdorff space such that each nonzero ring homomorphism from $C^*(X, \mathbb{R})$ into $\mathbb{R}$ has an obvious representation.

Let $I$ be the interval $[0, 1]$, provided with usual multiplication and unary mapping $x \to 1 - x$. From [2] and [3, Theorem 4.2] we infer, in the absence of the axiom of choice,

Proposition 1. A space $X$ is I-compact iff it is a completely regular Hausdorff space such that each morphism $H'$ from $C(X, I)$ into $I$ has an obvious representation.

Here "morphism" means a map that preserves multiplication and operation $x \to 1 - x$. I-compact spaces are spaces homeomorphic to closed subsets of products of $I$; even without the axiom of choice this class is productive and closed hereditary. So another proof of the compact*-version of Theorem 2 and Proposition 9 of [4] results from

Proposition 2. If $X$ is an arbitrary topological space, then the following are equivalent:
(1) Each nonzero ring homomorphism $C^*(X, \mathbb{R}) \to \mathbb{R}$ has an obvious representation.

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1Aspirant of the Belgian "Nationaal Fonds voor Wetenschappelijk Onderzoek."
Each morphism \( C(X, I) \to I \) has an obvious representation.

**Proof.** We show (1) \(\Rightarrow\) (2), the converse being easy. If \( H : C(X, I) \to I \) is a morphism, then \( H(a) = a \) whenever \( a \in I \) [2, Lemma 2]. Set \( \mathcal{D} = \{ f \in C^*(X, \mathbb{R}) : f > 1 \} \) and let \( H' : \mathcal{D} \to \mathbb{R} \) be defined by \( H'(f) = cH(f/c) \), where \( c > 0 \) and \( 0 < f/c < 1 \). This definition is unambiguous and does not depend on the axiom of choice since we may set \( c = \sup \{ f(x) : x \in X \} \). If \( f, g \in \mathcal{D} \), \( H'(fg) = H'(f)H'(g) \) by a routine argument. Furthermore we have
\[
H \left( f/(f + g) \right) = H \left( 1 - g/(f + g) \right) = 1 - H \left( g/(f + g) \right)
\]
so that \( 1 = H(f/(f + g)) + H(g/(f + g)) \). If we choose \( c > 0 \) such that \( 0 < (f + g)/c < 1 \), then
\[
H'(f + g) = cH((f + g)/c) = cH \left( f/c \right) + cH \left( g/c \right) = H'(f) + H'(g).
\]
Let \( H'' : C^*(X, \mathbb{R}) \to \mathbb{R} \) be defined by \( H''(f) = H'(f_1) - H'(f_2) \) if \( f = f_1 - f_2 \) with \( f_1, f_2 \in \mathcal{D} \). The definition is unambiguous and \( H''(f) = H'(1 + f + |f|) - H'(1 + |f|) \). An easy calculation shows that
\[
H''(f + g) = H''(f) + H''(g) \quad \text{and} \quad H''(f \cdot g) = H''(f) \cdot H''(g)
\]
for all \( f, g \in C^*(X, \mathbb{R}) \). If \( f \in C(X, I) \), then
\[
H''(f) = H'(f + 1) - H'(1) = 2H \left( (f + 1)/2 \right) - 2H \left( 1/2 \right)
\]
\[
= 2H \left( (f + 1)/2 \right) \left( H \left( f/(f + 1) \right) + H \left( 1/(f + 1) \right) \right) - 2H \left( 1/2 \right)
\]
\[
= 2H \left( f/2 \right) = H(f).
\]
Hence \( H'' \) is a nonzero ring homomorphism \( C^*(X, \mathbb{R}) \to \mathbb{R} \) and induces \( H \) on \( C(X, I) \); this completes the proof.

**References**


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