PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 66, Number 1, September 1977

THE CLASS OF COMPACT* SPACES IS PRODUCTIVE AND CLOSED HEREDITARY

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ABSTRACT. W. W. Comfort defined compact* spaces as completely regular Hausdorff spaces X such that each maximal ideal in the ring $C^*(X, \mathbf{R})$ of bounded continuous real-valued functions on X is fixed. He showed that, independently of the axiom of choice, the class of compact* spaces is productive and closed hereditary. We give a short new proof of this.

If X, E are topological spaces and $L \subseteq C(X, E)$, then a mapping H: $L \to E$ will be said to have an obvious representation iff there is a point $x_0 \in X$ such that $H(f) = f(x_0)$ for all $f \in L$.

In his Theorem 2.4* W. W. Comfort [1] proved that $C^*(X, \mathbf{R})/M$ is just the real field $\mathbf{R}$ whenever M is a maximal ideal of $C^*(X, \mathbf{R})$. So his Definition 3.1* is equivalent to the following

DEFINITION 1. A space X is compact* iff it is a completely regular Hausdorff space such that each nonzero ring homomorphism from $C^*(X, \mathbf{R})$ into $\mathbf{R}$ has an obvious representation.

Let I be the interval [0, 1], provided with usual multiplication and unary mapping $x \to 1 - x$. From [2] and [3, Theorem 4.2] we infer, in the absence of the axiom of choice,

PROPOSITION 1. *A space X is I-compact iff it is a completely regular Hausdorff space such that each morphism H' from $C(X, I)$ into I has an obvious representation.*

Here "morphism" means a map that preserves multiplication and operation $x \to 1 - x$. I-compact spaces are spaces homeomorphic to closed subsets of products of I; even without the axiom of choice this class is productive and closed hereditary. So another proof of the compact*-version of Theorem 2 and Proposition 9 of [4] results from

PROPOSITION 2. *If X is an arbitrary topological space, then the following are equivalent*:

(1) *Each nonzero ring homomorphism $C^*(X, \mathbf{R}) \to \mathbf{R}$ has an obvious representation.*

Received by the editors November 17, 1976 and, in revised form, April 3, 1977.
AMS (MOS) subject classifications (1970). Primary 54C30, 54D30; Secondary 04A25.
Key words and phrases. Compact* spaces, E-compactness, ring of continuous functions, representation of homomorphisms.

[1]Aspirant of the Belgian "Nationaal Fonds voor Wetenschappelijk Onderzoek."

(2) *Each morphism $C(X, I) \to I$ has an obvious representation.*

PROOF. We show (1) $\Rightarrow$ (2), the converse being easy. If $H: C(X, I) \to I$ is a morphism, then $H(a) = a$ whenever $a \in I$ [2, Lemma 2]. Set $\mathfrak{D} = \{f \in C^*(X, \mathbf{R}): f \geqslant 1\}$ and let $H': \mathfrak{D} \to \mathbf{R}$ be defined by $H'(f) = cH(f/c)$, where $c > 0$ and $0 \leqslant f/c \leqslant 1$. This definition is unambiguous and does not depend on the axiom of choice since we may set $c = \sup\{f(x): x \in X\}$. If $f, g \in \mathfrak{D}$, $H'(fg) = H'(f)H'(g)$ by a routine argument. Furthermore we have

$$H(f/(f+g)) = H(1 - g/(f+g)) = 1 - H(g/(f+g))$$

so that $1 = H(f/(f+g)) + H(g/(f+g))$. If we choose $c > 0$ such that $0 \leqslant (f+g)/c \leqslant 1$, then

$$\begin{aligned} H'(f+g) &= cH((f+g)/c) \\ &= cH((f+g)/c)\big(H(f/(f+g)) + H(g/(f+g))\big) \\ &= cH(f/c) + cH(g/c) = H'(f) + H'(g). \end{aligned}$$

Let $H'': C^*(X, \mathbf{R}) \to \mathbf{R}$ be defined by $H''(f) = H'(f_1) - H'(f_2)$ if $f = f_1 - f_2$ with $f_1, f_2 \in \mathfrak{D}$. The definition is unambiguous and $H''(f) = H'(1 + f + |f|) - H'(1 + |f|)$. An easy calculation shows that

$$H''(f+g) = H''(f) + H''(g) \quad \text{and} \quad H''(f \cdot g) = H''(f) \cdot H''(g)$$

for all $f, g \in C^*(X, \mathbf{R})$. If $f \in C(X, I)$, then

$$\begin{aligned} H''(f) &= H'(f+1) - H'(1) = 2H((f+1)/2) - 2H(1/2) \\ &= 2H((f+1)/2)\big(H(f/(f+1)) + H(1/(f+1))\big) - 2H(1/2) \\ &= 2H(f/2) = H(f). \end{aligned}$$

Hence H'' is a nonzero ring homomorphism $C^*(X, \mathbf{R}) \to \mathbf{R}$ and induces H on $C(X, I)$; this completes the proof.

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