A UNIQUE CONTINUATION THEOREM INVOLVING A DEGENERATE PARABOLIC OPERATOR

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Abstract. We consider the degenerate parabolic operator $L[u] = \gamma B[u] - u_t$ on a domain $D = \Omega \times (0, T]$ where $B[u] = \sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j}$ and $\gamma$ is an arbitrary complex number. Classically, $\gamma = 1$ and the real-valued matrix $(a_{ij})$ is positive definite. We assume $(a_{ij})$ is a real-valued symmetric matrix but not necessarily definite. We prove that any complex-valued function $u$ which satisfies the inequality $|L[u]| < c|u|$ for some nonnegative constant $c$ and vanishes initially as well as on the boundary of $\Omega$ must vanish on all of $D$. The theorem is particularly useful in studying uniqueness for many systems which are not parabolic.

1. Introduction. We consider the differential operator

(1.1) $L \equiv \gamma B - \partial / \partial t,$

where $\gamma$ is a complex number, and

(1.2) $B[u] \equiv \sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j},$

in which the real-valued matrix $(a_{ij}(x))$ defined on a domain $\Omega$ is assumed to be symmetric but not necessarily definite. We are using subscripts to denote differentiation.

A unique continuation theorem is established for the differential inequality

(1.3) $|L[u]|^2 \leq c |u|^2$

where $c$ is a nonnegative constant. Such results are useful in the study of uniqueness for systems of equations of the form (See [1].)

(1.4) $u_t^m = \sum_{i=1}^{N} \sum_{i,j=1}^{n}(a_{ij}(x)e^{im\Sigma_{j=1}^{n}u_{x_j}})_{x_i} + f_m(x, t, u)$

for $1 \leq m \leq N$. In the event that the constant matrix $(e^{im})$ has complex eigenvalues, classical results are not applicable to the system (1.4).

The problem of unique continuation for linear parabolic equations was considered by Lees and Protter [4] who generalized the results of several authors by considering differential inequalities of the form

(1.5) $(L[v])^2 \leq c_1 t^{-2}v^2 + c_2 \sum_{m=1}^{n}(v_{x_m})^2$

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in which it was assumed that \( \gamma = 1 \) in (1.1) and that the matrix \((a_{ij})\) be time dependent and satisfy the usual parabolicity condition of being positive definite.

We consider complex-valued functions \( u(x_1, \ldots, x_n, t) \) which satisfy (1.3) and which vanish initially as well as on the boundary of \( \Omega \). The proofs require considerably sharper integral estimates than those used by Lees and Protter [4] although the methods of the proofs are essentially the same.

2. The main results. Let \( D = \Omega \times (0, T] \) where \( \Omega \) is a bounded open connected set in real \( n \)-dimensional Euclidean space, \( \mathbb{R}^n \), with a sufficiently smooth boundary (\( \partial \Omega \)) to ensure the validity of integration by parts.

We shall say that the differential operator \( L \) defined in (1.1) and (1.2) satisfies condition \((G)\) if the real-valued symmetric matrix \((a_{ij})\) is continuous on \( \Omega \) with continuous bounded first order derivatives on \( \Omega \). The usual parabolicity condition that the matrix \((a_{ij})\) be (positive) definite is not required here. We shall use the notation

\[
(v, w) = \int_D v(x, t) \overline{w}(x, t) \, dx \, dt,
\]

\[
\|v\| = (v, v)^{1/2}.
\]

We let \( \tilde{P} \) be the set of complex-valued functions \( v \) such that \( v \) is continuous on \( \overline{D} \), the second partial derivatives of \( v \) in \( x \) are continuous and square integrable on \( D \), and the first partial derivatives of \( v \) in \( x \) are continuous and square integrable on \( D \). We let \( P \) be the set of functions in \( \tilde{P} \) which vanish on the boundary of \( D \). Finally we let \( \lambda(t) = t + \eta \) where \( \eta \) is any fixed positive number.

Before proving the main result of the paper, we need two preliminary lemmas.

**Lemma 1.** If the operator \( L \) satisfies condition \((G)\), \( u \in P, k \) is any positive integer and \( z(x, t) = [\lambda(t)]^{-k} u(x, t) \), then

\[
\|\gamma B[z]\|^2 + \|z\|^2 + k^2 \|\lambda^{-1}z\|^2 \geq 2 \text{Re}(\gamma B[z], z) + k \lambda^{-1}z).
\]

**Proof.** We let \( v = \text{Re}(z), w = \text{Im}(z), \alpha = \text{Re}(\gamma) \) and \( \beta = \text{Im}(\gamma) \). We now obtain

\[
\text{Re}(\gamma B[z], z_i) = \alpha(B[v], v_i) + \alpha(B[w], w_i)
\]

\[
+ \beta(B[w], w_i) - \beta(B[v], v_i).
\]

However, in view of the symmetry of \((a_{ij})\), we have

\[
(B[v], v_i) = - \sum_{i,j=1}^n (a_{ij} v_{x_i} v_{x_j}, v_{x_j})
\]

\[
= - \sum_{i,j=1}^n \int_D [a_{ij}/2] [v_{x_i} v_{x_j}]_t \, dx \, dt = 0.
\]

Similarly, we get
\((B[w], w) = 0.\)

Equation (2.4) combined with (2.5) and (2.6) yields
\[(2.7) \quad \text{Re}(\gamma B[z], z) = \beta (B[v], w) - \beta (B[w], v).
\]

We also have
\[(2.8) \quad \text{Re}(\gamma B[z], k\lambda^{-1}z) = \alpha (B[v], k\lambda^{-1}v) + \alpha (B[w], k\lambda^{-1}w) + \beta (B[v], k\lambda^{-1}v) - \beta (B[w], k\lambda^{-1}v).
\]

Since \(B\) is a symmetric operator, the last two terms in (2.8) add to zero. Thus (2.8) becomes
\[(2.9) \quad \text{Re}(\gamma B[z], k\lambda^{-1}z) = \alpha (B[v], k\lambda^{-1}v) + \alpha (B[w], k\lambda^{-1}w).
\]

With the use of (2.7) and (2.9), we get
\[
\begin{align*}
\|\gamma B[z]\|^2 + \|z\|^2 + \|k\lambda^{-1}z\|^2 - 2\text{Re}(\gamma B[z], z + k\lambda^{-1}z) \\
= (a^2 + \beta^2)\|B[z]\|^2 + \|z\|^2 + \|k\lambda^{-1}z\|^2 \\
- 2\beta (B[v], w) + 2\beta (B[w], v) - 2\alpha (B[v], k\lambda^{-1}v) \\
- 2\alpha (B[w], k\lambda^{-1}w)
\end{align*}
\]

\[(2.10) \quad = a^2\|B[v]\|^2 - 2\alpha (B[v], k\lambda^{-1}v) + \|k\lambda^{-1}v\|^2 \\
+ \beta^2\|B[v]\|^2 - 2\beta (B[v], w) + \|w\|^2 \\
+ \alpha^2\|B[w]\|^2 - 2\alpha (B[w], k\lambda^{-1}w) + \|k\lambda^{-1}w\|^2 \\
+ \beta^2\|B[w]\|^2 + 2\beta (B[w], v) + \|v\|^2
\]

Since the last term in (2.10) is nonnegative, this completes the proof.

**Lemma 2.** If the operator \(L\) satisfies condition (G), \(u \in P\), and \(k\) is any positive integer, then
\[(2.11) \quad \|\lambda^{-k}L[u]\|^2 > k\|\lambda^{-k}u\|^2.
\]

**Proof.** Let \(z(x, t) = [\lambda(t)]^{-k}u(x, t)\). Then \(z \in P\) and \(L[u] = L[\lambda^kz] = \gamma\lambda^kB[z] - \lambda^kz - k\lambda^{-1}z\). This identity yields
\[
\begin{align*}
\|\lambda^{-k}L[u]\|^2 = \|\gamma B[z] - z - k\lambda^{-1}z\|^2 \\
= \|\gamma B[z]\|^2 + \|z\|^2 + k^2\|\lambda^{-1}z\|^2 \\
- 2\text{Re}(\gamma B[z], z + k\lambda^{-1}z) + 2\text{Re}(k\lambda^{-1}z, z).
\end{align*}
\]

However, the last term in (2.12) may be written as
\[ 2\text{Re}(k\lambda^{-1}z, z_t) = k\int_D \lambda^{-1} |z|^2 \, dx \, dt = k\int \lambda^{-2} |z|^2 \, dx \, dt \]

(2.13)

\[ = k\|\lambda^{-1}z\|^2 = k\|\lambda^{-k-1}u\|^2. \]

Combining (2.12) and (2.13), we have

\[ \|\lambda^{-k}L[u]\|^2 - k\|\lambda^{-k-1}u\|^2 \]

(2.14)

\[ \geq \|\gamma B[z]\|^2 + \|z_t\|^2 + \|k\lambda^{-1}z\|^2 - 2\text{Re}(\gamma B[z], z_t + k\lambda^{-1}z). \]

From Lemma 1, we know that the right-hand side of equation (2.14) is nonnegative and this completes the proof.

**Theorem.** Suppose \( u \in \tilde{P} \), the operator \( L \) satisfies condition (G), and

\[ \|L[u]\|^2 \leq c\|u\|^2, \quad (x, t) \in D, \]

(2.15)

\[ u(x, 0) = 0, \quad x \in \Omega, \]

\[ u(s, t) = 0, \quad (s, t) \in \partial\Omega \times [0, T]. \]

Then \( u(x, t) = 0 \) for all \((x, t) \in D\).

**Proof.** Suppose \( t_1 \in (0, T) \). We wish to show \( u(x, t) = 0 \) for all \( t \in [0, t_1] \), \( x \in \Omega \). For this purpose choose \( 0 < t_1 < t_2 < t_3 < T \). Let \( \xi \) be a real-valued infinitely differentiable function defined on \([0, T]\) such that \( \xi(t) = 1 \) for \( t \in [0, t_2] \), \( \xi(t) = 0 \) for \( t \in [t_3, T] \), and \( 0 \leq \xi(t) \leq 1 \) for \( t \in [t_2, t_3] \). Now set \( v(x, t) = \xi(t)u(x, t) \) and observe that \( v \in P \). With the use of elementary methods and Lemma 2, we have

\[ (t_2 + \eta)^{-2} \int_{t_2}^{t_3} \int_{\Omega} \|L[v]\|^2 \, dx \, dt \geq \int_{t_2}^{t_3} \int_{\Omega} \lambda^{-2k}\|L[v]\|^2 \, dx \, dt \]

(2.16)

\[ = \|\lambda^{-k}L[v]\|^2 - \int_{0}^{t_2} \int_{\Omega} \lambda^{-2k}\|L[v]\|^2 \, dx \, dt \]

\[ > k\|\lambda^{-k-1}v\|^2 - c \int_{0}^{t_2} \int_{\Omega} \lambda^{-2k}\|v\|^2 \, dx \, dt \]

\[ > k \int_{0}^{t_2} \int_{\Omega} \lambda^{-2k-2}\|v\|^2 \, dx \, dt - c\eta^{-2} \int_{0}^{t_2} \int_{\Omega} \lambda^{-2k-2}\|v\|^2 \, dx \, dt \]

\[ = (k - c\eta^{-2}) \int_{0}^{t_2} \int_{\Omega} \lambda^{-2k-2}\|v\|^2 \, dx \, dt. \]

For \( k > c\eta^{-2} \), we obtain

\[ (k - c\eta^{-2}) \int_{0}^{t_2} \int_{\Omega} \lambda^{-2k-2}\|v\|^2 \, dx \, dt \]

(2.17)

\[ > (k - c\eta^{-2})(t_1 + \eta)^{-2k-2} \int_{0}^{t_1} \int_{\Omega} v^2 \, dx \, dt. \]

Combining (2.16) and (2.17) and simplifying, we get
\[
\left[ \frac{\eta^2}{(\eta^2 k - c)} \right] \left[ (t_1 + \eta) / (t_2 + \eta) \right]^{2k} \int_{t_2}^{t_1} \int_{\Omega} |L[v]|^2 \, dx \, dt
\]

\[\geq (t_1 + \eta)^{-2} \int_{0}^{t_1} \int_{\Omega} |v|^2 \, dx \, dt.
\]

Since \( t_1 < t_2 \), the left-hand side of (2.18) approaches zero as \( k \to \infty \) and this implies the right-hand side is zero since it is independent of \( k \). This completes the proof.

REFERENCES


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