A HELLY TYPE THEOREM ON THE SPHERE

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ABSTRACT. This paper establishes a Helly type theorem for convex sets in $S_n$, the $n$-dimensional unit sphere.

1. Introduction. Let $S_n$ denote the $n$-dimensional unit sphere in Euclidean $(n + 1)$-dimensional space, $R^{n+1}$.

A subset of $S_n$ will be called convex if it is the intersection of $S_n$ with a convex cone with apex 0 in $R^{n+1}$.

A subset of $S_n$ will be called strongly convex if it is convex and does not contain antipodal points.

Thus a set consisting of two antipodal points is convex but not strongly convex.

For other definitions of convex sets in $S_n$ and for a survey of Helly type theorems see [2]. For standard notation and terminology see [3].

M. J. C. Baker has proven in [1] the following

Theorem. For all positive integers $n$ and $t$, if the intersection of each $n + 1$ members of a family of at least $n + 1 + 2t$ strongly convex sets in $S_n$ is nonempty then the intersection of some $n + 1 + t$ members of the family is nonempty.

The purpose of this paper is to generalize Baker's result by proving

Theorem A. Let $n$ and $t$ be positive integers and let $\mathcal{C}$ be a family of $n + 1 + t$ convex sets in $S_n$.

If every $n + 1$ members of $\mathcal{C}$ have nonempty intersection then the intersection of some $n + 1 + [t/2]$ members of $\mathcal{C}$ is nonempty.

Moreover, $n + 1 + [t/2]$ cannot be replaced by $n + [t/2] + 2$.

It is possible to improve Theorem A for strongly convex sets and $t = 2$ by establishing

Theorem B. Let $n$ be an integer and let $\mathcal{C}$ be a family of $n + 3$ strongly convex sets in $S_n$. Suppose that every $n + 1$ members of $\mathcal{C}$ have nonempty intersection.

If $k$ is the number of different $n + 2$ membered subfamilies $\mathcal{B}$ of $\mathcal{C}$ such that $\bigcap \mathcal{B} = \emptyset$ then $k \leq 2$.
The proof of Theorem A is based on
(1) Properties of certain families of convex cones called nondegenerate families (N.D.F.'s), discussed in [4] and [5].
(2) The existence, or the nonexistence of certain \(k\)-neighborly polytopes in \(R^n\); see [3].
All the necessary definitions and properties of N.D.F.'s and of neighborly polytopes are stated in §2.
A proof of Theorem A is given in §3.
The proof of Theorem B is similar to the proof of Theorem A and is therefore omitted.

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2. Nondegenerate families and neighborly polytopes. Let \(\mathcal{A}\) be a nonempty finite family of convex cones with apex 0 in \(R^n\).
The family \(\mathcal{A}\) will be called a nondegenerate family (a N.D.F.) if each member of the family is of dimension \(n\), the intersection of any two members of the family is at least of dimension \((n - 1), \ldots\), the intersection of any \(n\)-members is at least of dimension 1 and the intersection of all members is \(\{0\}\).
A subset \(B\) of \(\mathcal{A}\) will be called a face of \(\mathcal{A}\) if \(\bigcap (\mathcal{A} \setminus B)\) is a subspace.
A subset \(B\) of \(\mathcal{A}\) is a \(k\)-face of \(\mathcal{A}\) if it is a face of \(\mathcal{A}\) and if \(\dim \bigcap (\mathcal{A} \setminus B) = |B| - k\). (\(|B|\) means the cardinality of \(B\).)
A relationship between N.D.F.'s and polytopes is given by

Theorem C. (i) If \(\mathcal{A}\) is a N.D.F. in \(R^n\) and \(|\mathcal{A}| = n + t\) then there exists a \((t - 1)\)-polytope \(P\) in \(R^{t-1}\) such that

(1) The lattice of faces of \(P\) and the lattice of faces of \(\mathcal{A}\), both ordered by the inclusion relation, are isomorphic. The isomorphism carries \(k\)-faces of \(P\) to \((k + 1)\)-faces of \(\mathcal{A}\).
(ii) For any \((t - 1)\)-polytope \(P\) with \(n + t\) vertices there exists a N.D.F. \(\mathcal{A}\) in \(R^n\) with \(|\mathcal{A}| = n + t\) such that (1) is satisfied.

A proof of Theorem C is given in [5].
Neighborly polytopes. A polytope \(P\) is \(k\)-neighborly if every subset of \(k\) vertices of \(P\) is the set of vertices of a proper face of \(P\).
Proofs of the following two theorems may be found in Chapter 7 of [3].

Theorem D. If \(P\) is a \(k\)-neighborly \(d\)-polytope then:
(i) All \(k\) vertices of \(P\) are affinely independent and \(P\) is \(k'\) neighborly for \(1 < k' < k\).
(ii) If \(k > \left[\frac{1}{2} d\right]\) then \(P\) is a simplex.

Theorem E. For every \(d\) and \(v > d\) there exist \(d\)-polytopes with \(v\) vertices which are \(\left[\frac{1}{2} d\right]\) neighborly.
3. Proof of Theorem A. In order to prove Theorem A it is sufficient to prove

**Theorem F.** Let $n$ and $t$ be nonnegative integers. Let $\mathcal{A}$ be a family of $n + t$ convex cones with apex 0 in $\mathbb{R}^n$ such that

\[(2) \bigcap \mathcal{A} = \{0\}\]

and

\[(3) \bigcap \mathcal{B} \neq \{0\} \text{ for any } n \text{ membered subfamily } \mathcal{B} \text{ of } \mathcal{A}.

Then there exists an $n + \lceil t/2 \rceil$ membered subfamily $\mathcal{C}$ of $\mathcal{A}$ such that $\bigcap \mathcal{C} \neq \{0\}$.

Moreover, $n + \lceil t/2 \rceil$ in the last statement cannot be replaced by $n + \lceil t/2 \rceil + 1$.

**Proof of Theorem F.** The proof is by induction on $n$. The cases $n = 0$ and $t < 1$ are trivial, so assume that $n > 0$ and that $t > 2$.

Let $\mathcal{A}$ be a family of convex cones with apex 0 on $\mathbb{R}^n$ such that $|\mathcal{A}| = n + t$ and assume that (2) and (3) are satisfied.

Suppose that $A$ is not a N.D.F. Let $\mathcal{H}$ be a maximal subset of $\mathcal{A}$ such that

\[(4) n' = \dim \bigcap \mathcal{H} < n - |\mathcal{H}|.

Clearly $\phi$ satisfies (4) since $\dim \bigcap \phi = n = n - |\phi|$.

It is not difficult to verify that $\dim \bigcap \mathcal{M} = n - |\mathcal{M}|$ and that $\mathcal{M} \neq \phi$ since $\mathcal{A}$ is not a N.D.F.

Define a family $\mathcal{B}' = \{ \bigcap \mathcal{M} \cap \{A\} : A \in \mathcal{A} \setminus \mathcal{H} \}$ of convex cones with apex 0 in $\mathbb{R}^n = \text{span} \bigcap \mathcal{H}$.

Since $|\mathcal{B}'| = n + t - |\mathcal{M}| = n' + t$ and since $\mathcal{B}'$ satisfies (2) and (3) we can use the induction hypothesis to obtain a $\mathcal{C}' \subset \mathcal{A} \setminus \mathcal{M}$ such that $|\mathcal{C}'| > n' + \lceil t/2 \rceil$ and $\bigcap \mathcal{C}' \neq \{0\}$.

Define $\mathcal{C} = \mathcal{C}' \cup \mathcal{M}$. The subfamily $\mathcal{C}$ has the desired properties since

\[|\mathcal{C}| = |\mathcal{C}'| + |\mathcal{M}| > n' + \lceil t/2 \rceil + |\mathcal{M}| = n + \lceil t/2 \rceil\]

and

\[\bigcap \mathcal{C} = \bigcap \mathcal{C}' \neq \{0\}.

Suppose that $A$ is a N.D.F. Assume by contradiction that $\bigcap \mathcal{B} = \{0\}$ for any $n + \lceil t/2 \rceil$ membered subfamily $\mathcal{B}$ of $\mathcal{A}$. Thus for any $\mathcal{C} \subset \mathcal{A}$ such that

\[|\mathcal{C}| \leq n + t - (n + \lceil t/2 \rceil) = t - \lceil t/2 \rceil,

$\mathcal{C}$ is a $|\mathcal{C}|$-face of $\mathcal{A}$.

Let $P$ be the $(t - 1)$-polytope described in Theorem C. It follows from (1) of Theorem C that $P$ is a $t - \lceil t/2 \rceil$ neighborholy polytope with $n + t$ vertices. Since $t - \lceil t/2 \rceil > \lceil (t - 1)/2 \rceil$, $P$ is by Theorem D a $(t - 1)$-simplex and therefore $n + t = (t - 1) + 1$, so that $n = 0$, a contradiction. This completes the proof of the first part of Theorem F.

To complete the proof of Theorem F let $P$ be a $\lceil (t - 1)/2 \rceil$ neighborly $(t - 1)$-polytope with $n + t$ vertices (see Theorem E).
Let $\mathcal{A}$ be the N.D.F. described in (ii) of Theorem C. The family satisfies (2) and (3) since any N.D.F. in $\mathbb{R}^n$ satisfies (2) and also (3) if $n > 0$.

For any $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \ni n + \lfloor t/2 \rfloor + 1$ we have that

$$|\mathcal{A} \setminus \mathcal{B}| < t - \lfloor t/2 \rfloor - 1 < \lfloor (t - 1)/2 \rfloor.$$ 

By Theorem C(i), for any such $\mathcal{B}$, $\mathcal{A} \setminus \mathcal{B}$ is a $|\mathcal{A} \setminus \mathcal{B}|$-face of $\mathcal{A}$.

Therefore $\dim(\cap \mathcal{B}) = \dim(\cap (\mathcal{A} \setminus (\mathcal{A} \setminus \mathcal{B}))) = |\mathcal{A} \setminus \mathcal{B}| - |\mathcal{A} \setminus \mathcal{B}| = 0$ so that $\cap \mathcal{B} = \{0\}$. The proof of Theorem F is now complete.

References


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