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A HELLY TYPE THEOREM ON THE SPHERE

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ABSTRACT. This paper establishes a Helly type theorem for convex sets in S_n, the n-dimensional unit sphere.

1. **Introduction.** Let S_n denote the n-dimensional unit sphere in Euclidean $(n+1)$-dimensional space, R^{n+1}.

A subset of S_n will be called *convex* if it is the intersection of S_n with a convex cone with apex 0 in R^{n+1}.

A subset of S_n will be called *strongly convex* if it is convex and does not contain antipodal points.

Thus a set consisting of two antipodal points is convex but not strongly convex.

For other definitions of convex sets in S_n and for a survey of Helly type theorems see [2]. For standard notation and terminology see [3].

M. J. C. Baker has proven in [1] the following

THEOREM. *For all positive integers n and t, if the intersection of each $n+1$ members of a family of at least $n+1+2t$ strongly convex sets in S_n is nonempty then the intersection of some $n+1+t$ members of the family is nonempty.*

The purpose of this paper is to generalize Baker's result by proving

THEOREM A. *Let n and t be positive integers and let $\mathcal{A}$ be a family of $n+1+t$ convex sets in S_n.*

If every $n+1$ members of $\mathcal{A}$ have nonempty intersection then the intersection of some $n+1+[t/2]$ members of $\mathcal{A}$ is nonempty.

Moreover, $n+1+[t/2]$ cannot be replaced by $n+[t/2]+2$.

It is possible to improve Theorem A for strongly convex sets and $t=2$ by establishing

THEOREM B. *Let n be an integer and let $\mathcal{A}$ be a family of $n+3$ strongly convex sets in S_n. Suppose that every $n+1$ members of $\mathcal{A}$ have nonempty intersection.*

If k is the number of different $n+2$ membered subfamilies $\mathcal{B}$ of $\mathcal{A}$ such that $\bigcap \mathcal{B} = \varnothing$ then $k \leqslant 2$.

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The proof of Theorem A is based on

(1) Properties of certain families of convex cones called *nondegenerate families* (*N.D.F.'s*), discussed in [4] and [5].

(2) The existence, or the nonexistence of certain k-neighborly polytopes in R^n; see [3].

All the necessary definitions and properties of N.D.F.'s and of neighborly polytopes are stated in §2.

A proof of Theorem A is given in §3.

The proof of Theorem B is similar to the proof of Theorem A and is therefore omitted.

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2. **Nondegenerate families and neighborly polytopes.** Let $\mathcal{A}$ be a nonempty finite family of convex cones with apex 0 in R^n.

The family $\mathcal{A}$ will be called a *nondegenerate family* (a *N.D.F.*) if each member of the family is of dimension n, the intersection of any two members of the family is at least of dimension $(n-1), \ldots,$ the intersection of any n-members is at least of dimension 1 and the intersection of all members is $\{0\}$.

A subset $\mathcal{B}$ of $\mathcal{A}$ will be called a *face of* $\mathcal{A}$ if $\bigcap(\mathcal{A} \setminus \mathcal{B})$ is a subspace.

A subset $\mathcal{B}$ of $\mathcal{A}$ is a *k-face* of $\mathcal{A}$ if it is a face of $\mathcal{A}$ and if $\dim \bigcap (\mathcal{A} \setminus \mathcal{B}) = |\mathcal{B}| - k$. ($|\mathcal{B}|$ means the cardinality of $\mathcal{B}$.)

A relationship between N.D.F.'s and polytopes is given by

THEOREM C. (i) *If $\mathcal{A}$ is a N.D.F. in R^n and $|\mathcal{A}| = n + t$ then there exists a $(t-1)$-polytope P in R^{t-1} such that*

(1) *The lattice of faces of P and the lattice of faces of $\mathcal{A}$, both ordered by the inclusion relation, are isomorphic. The isomorphism carries k-faces of P to $(k+1)$-faces of $\mathcal{A}$.*

(ii) *For any $(t-1)$-polytope P with $n + t$ vertices there exists a N.D.F. $\mathcal{A}$ in R^n with $|\mathcal{A}| = n + t$ such that* (1) *is satisfied.*

A proof of Theorem C is given in [5].

NEIGHBORLY POLYTOPES. A polytope P is k-neighborly if every subset of k vertices of P is the set of vertices of a proper face of P.

Proofs of the following two theorems may be found in Chapter 7 of [3].

THEOREM D. *If P is a k-neighborly d-polytope then*:

(i) *All k vertices of P are affinely independent and P is k' neighborly for $1 \leq k' \leq k$.*

(ii) *if $k > [\frac{1}{2}d]$ then p is a simplex.*

THEOREM E. *For every d and $v > d$ there exist d-polytopes with v vertices which are $[\frac{1}{2}d]$ neighborly.*

3. **Proof of Theorem** A. In order to prove Theorem A it is sufficient to prove

THEOREM F. *Let n and t be nonnegative integers. Let $\mathcal{A}$ be a family of $n + t$ convex cones with apex 0 in R^n such that*

$$\bigcap \mathcal{A} = \{0\} \tag{2}$$

and

$$\bigcap \mathcal{B} \neq \{0\} \quad \textit{for any } n \textit{ membered subfamily } \mathcal{B} \textit{ of } \mathcal{A}. \tag{3}$$

Then there exists an $n + [t/2]$ membered subfamily $\mathcal{C}$ of $\mathcal{A}$ such that $\bigcap \mathcal{C} \neq \{0\}$.

Moreover, $n + [t/2]$ in the last statement cannot be replaced by $n + [t/2] + 1$.

PROOF OF THEOREM F. The proof is by induction on n. The cases $n = 0$ and $t \leqslant 1$ are trivial, so assume that $n > 0$ and that $t \geqslant 2$.

Let $\mathcal{A}$ be a family of convex cones with apex 0 on R^n such that $|\mathcal{A}| = n + t$ and assume that (2) and (3) are satisfied.

Suppose that A is not a N.D.F. let $\mathcal{M}$ be a maximal subset of $\mathcal{A}$ such that

$$n' = \dim \bigcap \mathcal{M} \leqslant n - |\mathcal{M}|. \tag{4}$$

Clearly ϕ satisfies (4) since $\dim \bigcap \phi = n = n - |\phi|$.

It is not difficult to verify that $\dim \bigcap \mathcal{M} = n - |\mathcal{M}|$ and that $\mathcal{M} \neq \phi$ since $\mathcal{A}$ is not a N.D.F.

Define a family $\mathcal{A}' = \{\bigcap \mathcal{M} \cap \{A\}: A \in \mathcal{A} \setminus \mathcal{M}\}$ of convex cones with apex 0 in $R^{n'} = \operatorname{span} \bigcap \mathcal{M}$.

Since $|\mathcal{A}'| = n + t - |\mathcal{M}| = n' + t$ and since $\mathcal{A}'$ satisfies (2) and (3) we can use the induction hypothesis to obtain a $\mathcal{C}' \subset \mathcal{A} \setminus \mathcal{M}$ such that $|\mathcal{C}'| \geqslant n' + [t/2]$ and $\bigcap \mathcal{C}' \neq \{0\}$.

Define $\mathcal{C} = \mathcal{C}' \cup \mathcal{M}$. The subfamily $\mathcal{C}$ has the desired properties since

$$|\mathcal{C}| = |\mathcal{C}'| + |\mathcal{M}| \geqslant n' + [t/2] + |\mathcal{M}| = n + [t/2]$$

and

$$\bigcap \mathcal{C} = \bigcap \mathcal{C}' \neq \{0\}.$$

Suppose that A is a N.D.F. Assume by contradiction that $\bigcap \mathcal{B} = \{0\}$ for any $n + [t/2]$ membered subfamily $\mathcal{B}$ of $\mathcal{A}$. Thus for any $\mathcal{C} \subset \mathcal{A}$ such that

$$|\mathcal{C}| \leqslant n + t - (n + [t/2]) = t - [t/2],$$

$\mathcal{C}$ is a $|\mathcal{C}|$-face of $\mathcal{A}$.

Let P be the $(t - 1)$-polytope described in Theorem C. It follows from (1) of Theorem C that P is a $t - [t/2]$ neighborholy polytope with $n + t$ vertices. Since $t - [t/2] > [(t - 1)/2]$, P is by Theorem D a $(t - 1)$-simplex and therefore $n + t = (t - 1) + 1$, so that $n = 0$, a contradiction. This completes the proof of the first part of Theorem F.

To complete the proof of Theorem F let P be a $[(t - 1)/2]$ neighborly $(t - 1)$-polytope with $n + t$ vertices (see Theorem E).

Let $\mathcal{A}$ be the N.D.F. described in (ii) of Theorem C. The family satisfies (2) and (3) since any N.D.F. in R^n satisfies (2) and also (3) if $n > 0$.

For any $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B} \geqslant n + [t/2] + 1$ we have that

$$|\mathcal{A} \setminus \mathcal{B}| \leqslant t - [t/2] - 1 \leqslant [(t-1)/2].$$

By Theorem C(i), for any such $\mathcal{B}$, $\mathcal{A} \setminus \mathcal{B}$ is a $|\mathcal{A} \setminus \mathcal{B}|$-face of $\mathcal{A}$.

Therefore $\dim \bigcap \mathcal{B} = \dim \bigcap (\mathcal{A} \setminus (\mathcal{A} \setminus \mathcal{B})) = |\mathcal{A} \setminus \mathcal{B}| - |\mathcal{A} \setminus \mathcal{B}| = 0$ so that $\bigcap \mathcal{B} = \{0\}$. The proof of Theorem F is now complete.

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