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# THE RANGE OF A VECTOR MEASURE HAS THE BANACH-SAKS PROPERTY[1]

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ABSTRACT. The above result of Diestel and Seifert is proved using the Banach-Saks theorem for $L^p(\lambda)$, $1 < p < \infty$.

Let $X$ be a real Hausdorff quasi-complete locally convex space (L.C.S.), $\mathcal{Q}$ be a $\alpha$-algebra of subsets of some set $S$ and $\nu$: $\mathcal{Q} \to X$ be a measure. We assume that $\nu$ is controlled, viz. that there exists a finite positive measure $\lambda$ on $\mathcal{Q}$ such that $\nu$ and $\lambda$ have the same null sets. When $X$ is a Banach space, Bartle, Dunford and Schwartz [3] proved every measure $\nu$: $\mathcal{Q} \to X$ to be controlled. This has been extended to a class of L.C.S. $X$, including metrizable ones, by Tweddle [8].

A set $A \subset X$ is said to have the *Banach-Saks property* if every sequence in $A$ has a subsequence $\{x_n\}$ such that the sequence of arithmetic means formed from $\{x_n\}$ converges. Using the methods of Szlenk [6], Diestel and Seifert [5] recently proved that $K = \overline{\text{co}}\, \nu(\mathcal{Q})$ has the Banach-Saks property when $X$ is a Banach space. Our proof of an extension of this result is based instead on the Banach-Saks theorem for $L^p(\lambda)$, $1 < p < \infty$ [2], [4, p. 78].

Since $X$ is quasi-complete, the set $K = \overline{\text{co}}\, \nu(\mathcal{Q})$ is weakly compact by Theorem 3 of Tweddle [7]. Further, for each $\phi \in L^\infty(\lambda)$ the weak integral $\int \phi \, d\nu$ belongs to $X$ (see e.g. [1, Lemma 1]). Let $T$: $L^\infty(\lambda) \to X$ be the weak integral map, and let $T_0$ denote its restriction to the weak*-compact convex set

$$P \equiv \{\phi \in L^\infty(\lambda) \colon 0 \leqslant \phi(s) \leqslant 1 \ s\text{-a.e.}\}.$$

Then $T_0(P) = K$.

As in Proposition 1 of [1] (proved there for $p = 1$), the next result easily follows from Egoroff's theorem.

1. PROPOSITION. *Let $1 < p < \infty$. Then the restriction of $T$ to every norm bounded subset $B$ of $L^\infty(\lambda)$ is continuous relative to the $L^p$-norm on $B$ and the given topology of $X$.*

2. THEOREM. *Let $\nu$: $\mathcal{Q} \to X$ be a controlled measure. Then $K = \overline{\text{co}}\, \nu(\mathcal{Q})$ has the Banach-Saks property.*

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PROOF. Let $\{x_n\}$ be a sequence in $K$. Since $K$ is weakly compact, by Eberlein's theorem we may replace $\{x_n\}$ by a subsequence $\{x_m\}$ that converges weakly to some $x \in K$. Since $T(P) = K$, for every $m$ there exists $\phi_m \in P$ such that $T(\phi_m) = x_m$. Fix $p$, $1 < p < \infty$. As $P$ is a bounded subset of $L^p(\lambda)$, by the Banach-Saks theorem there exists a subsequence $\{\phi_i\}$ of $\{\phi_m\}$ such that the sequence $\{\psi_i\}$ formed of arithmetic means of $\{\phi_i\}$ converges to some $\psi \in L^p(\lambda)$ in the norm of $L^p$. Clearly, $\psi \in P$. By Proposition 1 and the linearity of $T$ the sequence $\{T(\phi_m)\}$, viz. $\{x_m\}$ has arithmetic means that converge to $T(\psi) \in T(P) = K$. Since $\{x_m\}$ converges weakly to $x$, we have $T(\psi) = x$, which completes the proof.

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