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STICKY ARCS IN E^n $(n \geqslant 4)$

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ABSTRACT. Let A and B be arcs in E^3, Euclidean 3-space. Then A can be "slipped" off B; i.e., there exists a homeomorphism of E^3 onto itself, arbitrarily close to the identity, such that $h(A) \cap B = \varnothing$. The purpose of this note is to show that arcs in E^n $(n \geqslant 4)$ do not always enjoy this property. The examples depend heavily on a recent result of McMillan.

1. Introduction. If X and Y are subsets of E^n, Euclidean n-space, we say that X can be *slipped* off Y in E^n if for each $\varepsilon > 0$ there is an ε-homeomorphism $h: E^n \to E^n$ such that $h(X) \cap Y = \varnothing$; otherwise, we say X *cannot be slipped off* Y. Results of Armentrout [1] and McMillan [5] show that if A and B are arcs in E^3, then A can be slipped off B. We show that this is false in higher dimensions by proving the following

THEOREM. *There exist cellular arcs A and B in E^n $(n \geqslant 4)$ such that A cannot be slipped off B.*

Existence of A and B. McMillan [4] has shown, for each $n \geqslant 4$, there exists an arc C in an n-manifold M^n which has no neighborhood in M^n which embeds in E^n. Without loss of generality, we may assume that C is the union of arcs A' and B' whose intersection is a common endpoint and such that A' and B' have neighborhoods U and V, respectively, which embed in E^n. Let $f: U \to E^n$ and $g: V \to E^n$ be embeddings. By [2] and a correct choice of U and V, we may assume that f and g agree on $U \cap V$. We now let $A = f(A')$ and $B = g(B')$.

PROOF OF THEOREM. Let B_1 and B_2 be concentric n-balls centered at $f(A' \cap B')$ such that $B_1 \subset \operatorname{Int} B_2 \subset f(U \cap V) = g(U \cap V)$. We suppose that A can be slipped off B and choose a homeomorphism $h: E^n \to E^n$ such that $h(A) \cap B = \varnothing$. Furthermore, h is chosen so close to the identity that the following conditions are satisfied.

(1) $h(\operatorname{Bd} B_2) \cap B_1 = \varnothing$.

(2) There is an embedding $e: B_2 - \operatorname{Int} B_1 \to E^n$ such that $e|\operatorname{Bd} B_1 =$ identity and $e|\operatorname{Bd} B_2 = h|\operatorname{Bd} B_2$.

(3) The embedding e is so close to the identity that $e(A \cap (B_2 - \operatorname{Int} B_1)) \cap B = \varnothing$.

The existence of the embedding e is a corollary of the Kirby-Edwards local contractibility theorems [3] and is used explicitly in [6].

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Consider the homeomorphism H: $E^n \to E^n$ given by $H|B_1$ = identity, $H|(B_2 - \operatorname{Int} B_1) = e$, and $H|(E^n - B_2) = h|(E^n - B_2)$. Thus, $H \circ f$: $U \to E^n$ and g: $V \to E^n$ are embeddings which agree on a neighborhood of $A' \cap B'$ in M^n and such that $H \circ f(A') \cap g(B')$ is a single point. By choosing even smaller neighborhoods U' and V' of A' and B', respectively, we may assume that $f' = H \circ f|U'$ and $g' = g|V'$ agree on $U' \cap V'$ and that $f'(U') \cap g'(V') = f'(U' \cap V') = g'(U' \cap V')$. Thus, f' and g' can be used to give an embedding of $U' \cup V'$ into E^n which yields a contradiction.

An examination of McMillan's arc C shows that the interior of C has a neighborhood in M^n which is homeomorphic with the suspension of a set X, ΣX, the homeomorphism sending C to $\Sigma\{x\}$, $x \in X$. The set X is obtained from a piecewise-linear $(n - 1)$-manifold by identifying a cell-like set in its interior to the point x. This neighborhood may be used to show that any proper subarc of C can be pushed arbitrarily close to one of its endpoints by a homeomorphism of M^n which has support in an arbitrary neighborhood in M^n of the subarc. This fact implies that any subarc of C is cellular. Therefore we may assume that A and B are cellular.

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