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FINITE SIMPLE GROUPS CONTAINING A SELF-CENTRALIZING ELEMENT OF ORDER 6

JOHN L. HAYDEN AND DAVID L. WINTER

ABSTRACT. By a self-centralizing element of a group we mean an element which commutes only with its powers. In this paper we establish the following result:

THEOREM. *Let G be a finite simple group which has a self-centralizing element of order* 6. *Assume that G has only one class of involutions. Then G is isomorphic to one of the groups* M_{11}, J_1, $L_3(3)$, $L_2(11)$, $L_2(13)$.

By a self-centralizing (s.c.) element of a group we mean an element which commutes only with its own powers. The structure of a finite group containing a s.c. element of order 2 has been known for a long time. The classification of finite groups with a s.c. element of order 3 was carried out by Feit and Thompson [2] and is an important tool in this paper. The determination of finite simple groups containing a s.c. element of prime order $p > 3$ is a classical problem as yet unsolved although some special cases have been handled. In [12], M. Suzuki proved that the only finite simple groups containing a s.c. element of order 4 are $L_2(7)$, A_6 and A_7. Finite simple groups containing a s.c. element of order 8 are studied in [10]. In this paper we establish the following result.

THEOREM. *Let G be a finite simple group which has a self-centralizing element of order* 6. *Assume that G has only one class of involutions. Then G is isomorphic to one of the groups* M_{11}, J_1, $L_3(3)$, $L_2(11)$, $L_2(13)$.

The alternating groups A_8, A_9 are examples of simple groups with a s.c. element of order 6 which has two classes of involutions. As indicated in [8], it may be very difficult to handle groups containing a s.c. element of order 6 having more than one class of involutions.

The smallest Ree group is an example of a nonsimple group containing a s.c. element of order 6 and having only one class of involutions. It is isomorphic to an extension of $L_2(2^3)$ by a field automorphism of order 3.

We shall make use of the following theorem.

(1.1) ([2]). *If Y is a finite group with a s.c. element of order* 3, *then Y has a normal subgroup N such that one of the following holds*:

(i) Y/N *is cyclic of order* 3 *and N is nilpotent*.

(ii) Y/N *is dihedral of order* 6 *and N is nilpotent*.

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(iii) $Y/N \cong A_5$ *and* N *is an elementary abelian* 2-*group*.
(iv) $Y/N \cong L_2(7)$ *and* $N = 1$.

From now on we shall let G denote a group satisfying the hypotheses of our theorem. Then G contains an element x of order 3 and an involution t such that $[t, x] = 1$ and $C(tx) = \langle tx \rangle$. Furthermore every involution of G is conjugate to t.

(1.2) *$C(x) = \langle t \rangle L$ where L is a nilpotent normal subgroup of $C(x)$ of odd order. $L/\langle x \rangle$ is abelian and is inverted by t.*

PROOF. Let $\bar{C} = C(x)/\langle x \rangle$ and let $\bar{v} \in \bar{C}$ be such that $[\bar{v}, \bar{t}] = \bar{1}$. Then $\overline{[v, t]} = \bar{1}$ and so $[v, t] = x_0 \in \langle x \rangle$. Hence $v^{-1}tv = x_0 t$. Since t has order 2, $x_0 = 1$ and $v \in C(t) \cap C(x) = \langle xt \rangle$. Hence $\bar{v} \in \langle \bar{t} \rangle$. Thus $\bar{t}$ is a s.c. element of $\bar{C}$ of order 2. The structure of $\bar{C}$ is therefore known and (1.2) follows easily.

(1.3) *Let H be a subgroup of G with $tx \in H$. Assume that H contains a normal 2-subgroup U with $t \in U$. Then $C_H(x) = \langle tx \rangle$. Also if $\bar{H} = H/U$, then $\bar{x}$ is a s.c. element of $\bar{H}$ of order 3.*

PROOF. By (1.2), $U \cap C(x) = \langle t \rangle \lhd C_H(x)$. But by (1.2), $N_{C(x)}(\langle t \rangle) = \langle tx \rangle$. Hence $C_H(x) = \langle tx \rangle$. This implies that $\langle x \rangle$ is a Sylow 3-subgroup of H. It is well known and easily verified that $\overline{N(\langle x \rangle)} = N_{\bar{H}}(\langle \bar{x} \rangle)$. It is also easily checked from this that $\bar{y} \in \bar{H}$ centralizes $\bar{x}$ if and only if $y \in C_H(x)U$. Therefore $C_{\bar{H}}(\bar{x}) = \overline{\langle tx \rangle U} = \langle \bar{x} \rangle$.

By setting $H = C(t)$, $U = \langle t \rangle$, we have as an immediate consequence of (1.1) and (1.3)

(1.4) *$C(t)$ contains a normal subgroup $N \geqslant \langle t \rangle$ such that one of the following holds*:
(i) *$C(t)/N$ is cyclic of order 3 and $N/\langle t \rangle$ is nilpotent.*
(ii) *$C(t)/N$ is dihedral of order 6 and N is nilpotent.*
(iii) *$C(t)/N \cong A_5$ and $N/\langle t \rangle$ is an elementary abelian 2-group.*
(iv) *$C(t)/N \cong L_2(7)$ and $N = \langle t \rangle$.*

The remainder of the proof of the theorem is divided into the four cases determined by (1.4). Assume first that (1.4) (iv) holds. Then $C(t)/\langle t \rangle \cong L_2(7)$. It has been proved by Schur [11, p. 120], that $C(t)$ is either isomorphic to a direct product of $\langle t \rangle$ and $L_2(7)$ or to $SL(2,7)$. By a result of Janko and Thompson [9], the first case is not possible. In the second case $C(t)$, and hence G, has a quaternion Sylow 2-subgroup. By a well-known result of Brauer and Suzuki, G cannot be simple. Therefore (1.4) (iv) cannot hold.

Assume next that (1.4) (iii) holds with $N = \langle t \rangle$. The argument of the preceding paragraph shows $C(t) = \langle t \rangle \times K$, $K \cong A_5$. Therefore, by [7] G is isomorphic to J_1.

In the remaining cases $C(t)$ is solvable or $C(t)/N \cong A_5$. In the latter case $|N| > 2$ with $C(N) \subseteq N$. Hence $C(t)$ is 2-constrained in all cases. As G has only one class of involutions, a theorem of Gorenstein and Goldschmidt [3, p. 74] implies $O(C(t)) = 1$ or $SCN_3(2) = \varnothing$. In the latter case, Corollary 4, [5] shows G is isomorphic to one of the groups $L_2(q)$, $L_3(q)$, $U_3(q)$, q odd, $U_3(4)$, A_7 or M_{11}. The only groups among these satisfying our hypotheses are $L_3(3)$, M_{11}, $L_2(11)$ and $L_2(13)$.

We may now assume $O(C(t)) = 1$. This forces N to be a 2-group and $|C(t)| = 2^a \cdot 3 \cdot 5^b$ for some a, b. We now claim that if M is a 2-local subgroup of G, then each Sylow subgroup of M of odd order is cyclic. Without loss we may assume $t \in Z(O_2(M)) = Z$. Suppose M contains an elementary abelian p-subgroup P of order p^2, p odd. By a well-known result [4, p. 188], $Z = \Pi_{y \in P^*} C_Z(y)$. Again without loss, we can assume $t \in C_Z(y)$ for some $y \in P^\#$. Then $y \in C(t)$, $p \in \{3, 5\}$. Now let $y_1 \in P - \langle y \rangle$. Since $Z \lhd M$, $t^{y_1} \in Z \cap C(t)$. Also, $(t^{y_1})^y = (t^y)^{y_1} = t^{y_1}$ so $t^{y_1} \in C(t) \cap C(y)$. If $p = 3$, $|y| = 3$ and, as $\langle ty \rangle$ is conjugate to $\langle tx \rangle$ in $C(t)$, $C(t) \cap C(y) = \langle ty \rangle$. If $p = 5$, we are in case (1.4)(iii). Let $\bar{C} = C(t)/\langle t \rangle$. As $\bar{x}$ is a s.c. element of order 3 in $\bar{C}$, Theorem 8.2 of [6] implies $\bar{N} = N/\langle t \rangle$ is the direct sum of minimal normal subgroups of $\bar{C}$ of order 16 on which $A_5 \cong SL(2, 4)$ acts in the natural way. Consequently $\bar{y}$ is a s.c. element of order 5 in $\bar{C}$ and $C(t) \cap C(y) = \langle ty \rangle$ in this case as well. In all cases $t^{y_1} \in C(t) \cap C(y) = \langle ty \rangle$ so $t^{y_1} = t$. Hence $y_1 \in C(ty) = \langle ty \rangle$ and $y_1 \in \langle y \rangle$, a contradiction proving the claim. A recent result of M. Aschbacher [1] now gives the possibilities for G. We see that there is no additional group satisfying our conditions. This completes the proof of the theorem.

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Department of Mathematics, Bowling Green State University, Bowling Green, Ohio 43403

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824