FINITE SIMPLE GROUPS CONTAINING
A SELF-CENTRALIZING ELEMENT OF ORDER 6

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Abstract. By a self-centralizing element of a group we mean an element
which commutes only with its powers. In this paper we establish the
following result:

Theorem. Let $G$ be a finite simple group which has a self-centralizing
element of order 6. Assume that $G$ has only one class of involutions. Then $G$ is
isomorphic to one of the groups $M_{11}, J_1, L_3(3), L_2(11), L_2(13)$.

By a self-centralizing (s.c.) element of a group we mean an element which
commutes only with its own powers. The structure of a finite group
containing a s.c. element of order 2 has been known for a long time. The
classification of finite groups with a s.c. element of order 3 was carried out by
Feit and Thompson [2] and is an important tool in this paper. The
determination of finite simple groups containing a s.c. element of prime order
$p > 3$ is a classical problem as yet unsolved although some special cases have
been handled. In [12], M. Suzuki proved that the only finite simple groups
containing a s.c. element of order 4 are $L_2(7), A_6$ and $A_7$. Finite simple groups
containing a s.c. element of order 8 are studied in [10]. In this paper we
establish the following result.

Theorem. Let $G$ be a finite simple group which has a self-centralizing
element of order 6. Assume that $G$ has only one class of involutions. Then $G$ is
isomorphic to one of the groups $M_{11}, J_1, L_3(3), L_2(11), L_2(13)$.

The alternating groups $A_8, A_9$ are examples of simple groups with a s.c.
element of order 6 which has two classes of involutions. As indicated in [8], it
may be very difficult to handle groups containing a s.c. element of order 6
having more than one class of involutions.

The smallest Ree group is an example of a nonsimple group containing a
s.c. element of order 6 and having only one class of involutions. It is
isomorphic to an extension of $L_2(2^3)$ by a field automorphism of order 3.

We shall make use of the following theorem.

(1.1) ([2]). If $Y$ is a finite group with a s.c. element of order 3, then $Y$ has a
normal subgroup $N$ such that one of the following holds:

(i) $Y/N$ is cyclic of order 3 and $N$ is nilpotent.
(ii) $Y/N$ is dihedral of order 6 and $N$ is nilpotent.

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(iii) \( Y/N \cong A_5 \) and \( N \) is an elementary abelian 2-group.
(iv) \( Y/N \cong L_2(7) \) and \( N = 1 \).

From now on we shall let \( G \) denote a group satisfying the hypotheses of our theorem. Then \( G \) contains an element \( x \) of order 3 and an involution \( t \) such that \([t, x] = 1\) and \( C(tx) = \langle tx \rangle\). Furthermore every involution of \( G \) is conjugate to \( t \).

(1.2) \( C(x) = \langle t \rangle L \) where \( L \) is a nilpotent normal subgroup of \( C(x) \) of odd order. \( L/\langle x \rangle \) is abelian and is inverted by \( t \).

**Proof.** Let \( C = C(x)/(x) \) and let \( v \in C \) be such that \([v, t] = 1\). Then \([v, t] = 1\) and so \([v, t] = x_0 \in \langle x \rangle\). Hence \( v^{-1}tv = x_0 t \). Since \( t \) has order 2, \( x_0 = 1 \) and \( v \in C(t) \cap C(x) = \langle x \rangle \). Hence \( v \in \langle t \rangle \). Thus \( t \) is a s.c. element of \( C \) of order 2. The structure of \( C \) is therefore known and (1.2) follows easily.

(1.3) Let \( H \) be a subgroup of \( G \) with \( tx \in H \). Assume that \( H \) contains a normal 2-subgroup \( U \) with \( t \in U \). Then \( CH(x) = \langle tx \rangle \). Also if \( \tilde{H} = H/U \), then \( \tilde{x} \) is a s.c. element of \( H \) of order 3.

**Proof.** By (1.2), \( U \cap C(x) = \langle x \rangle \cap CH(x) \). But by (1.2), \( N_{CH(x)}(\langle t \rangle) = \langle tx \rangle \). Hence \( CH(x) = \langle tx \rangle \). This implies that \( \langle x \rangle \) is a Sylow 3-subgroup of \( H \). It is well known and easily verified that \( N(\langle x \rangle) = N_{\tilde{H}}(\langle \tilde{x} \rangle) \). It is also easily checked from this that \( y \in \tilde{H} \) centralizes \( \tilde{x} \) if and only if \( y \in CH(x)U \). Therefore \( CH(x) = \langle tx \rangle \).

(1.4) \( C(t) \) contains a normal subgroup \( N \supset \langle t \rangle \) such that one of the following holds:
(i) \( C(t)/N \) is cyclic of order 3 and \( N/\langle t \rangle \) is nilpotent.
(ii) \( C(t)/N \) is dihedral of order 6 and \( N \) is nilpotent.
(iii) \( C(t)/N \cong A_5 \) and \( N/\langle t \rangle \) is an elementary abelian 2-group.
(iv) \( C(t)/N \cong L_2(7) \) and \( N = \langle t \rangle \).

The remainder of the proof of the theorem is divided into the four cases determined by (1.4). Assume first that (1.4) (iv) holds. Then \( C(t)/\langle t \rangle \cong L_2(7) \). It has been proved by Schur [11, p. 120], that \( C(t) \) is either isomorphic to a direct product of \( \langle t \rangle \) and \( L_2(7) \) or to \( SL(2,7) \). By a result of Janko and Thompson [9], the first case is not possible. In the second case \( C(t) \), and hence \( G \), has a quaternion Sylow 2-subgroup. By a well-known result of Brauer and Suzuki, \( G \) cannot be simple. Therefore (1.4) (iv) cannot hold.

Assume next that (1.4) (iii) holds with \( N = \langle t \rangle \). The argument of the preceding paragraph shows \( C(t) = \langle t \rangle \times K, K \cong A_5 \). Therefore, by [7] \( G \) is isomorphic to \( J_1 \).

In the remaining cases \( C(t) \) is solvable or \( C(t)/N \cong A_5 \). In the latter case \( |N| > 2 \) with \( C(N) \subseteq N \). Hence \( C(t) \) is 2-constrained in all cases. As \( G \) has
only one class of involutions, a theorem of Gorenstein and Goldschmidt [3, p. 74] implies \( O(C(t)) = 1 \) or \( SCN_3(2) = \emptyset \). In the latter case, Corollary 4, [5] shows \( G \) is isomorphic to one of the groups \( L_2(q), L_3(q), U_3(q), q \) odd, \( U_4(4), \) \( A_n \), or \( M_{11} \). The only groups among these satisfying our hypotheses are \( L_3(3), M_{11}, L_3(11) \) and \( L_3(13) \).

We may now assume \( O(C(t)) = 1 \). This forces \( N \) to be a 2-group and \( |C(t)| = 2^a \cdot 3^b \cdot 5^c \) for some \( a, b \). We now claim that if \( M \) is a 2-local subgroup of \( G \), then each Sylow subgroup of \( M \) of odd order is cyclic.

Without loss we may assume \( t \in Z(O_2(M)) = Z \). Suppose \( M \) contains an elementary abelian \( p \)-subgroup \( P \) of order \( p \). Without loss, we can assume \( t \in Z \). Then \( y \in C(t) \), \( p \in \{3, 5\} \). Now let \( y_1 \in P - \langle y \rangle \). Since \( P < M \), \( t^{y_1} \in Z \cap C(t) \). Also, \( (t^{y_1})^y = (t^y)^{y_1} = t^{y_1} \) so \( t^{y_1} \in C(t) \cap C(y) \). If \( p = 3, |y| = 3 \) and, as \( \langle ty \rangle \) is conjugate to \( \langle tx \rangle \) in \( C(t) \), \( C(t) \cap C(y) = \langle ty \rangle \).

If \( p = 5 \), we are in case (1.4)(iii). Let \( \overline{C} = C(t)/\langle ty \rangle \). As \( \overline{y} \) is a s.c. element of order 3 in \( \overline{C} \), Theorem 8.2 of [6] implies \( \overline{N} \cong N/\langle t \rangle \) is the direct sum of minimal normal subgroups of \( \overline{C} \) of order 16 on which \( A_5 \cong SL(2, 4) \) acts in the natural way. Consequently \( \overline{y} \) is a s.c. element of order 5 in \( \overline{C} \) and \( C(t) \cap C(y) = \langle ty \rangle \) in this case as well. In all cases \( t^{y_1} \in C(t) \cap C(y) = \langle ty \rangle \) so \( t^{y_1} = t \). Hence \( y_1 \in C(ty) = \langle ty \rangle \) and \( y_1 \in \langle ty \rangle \), a contradiction proving the claim. A recent result of M. Aschbacher [1] now gives the possibilities for \( G \). We see that there is no additional group satisfying our conditions. This completes the proof of the theorem.

References

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