ON THE NONEXISTENCE OF HYPERCOMMUTING POLYNOMIALS

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Abstract. If \( f(x_1, \ldots, x_n) \) is not central for \( R \), then the additive group generated by all specializations of \( f \) in \( R \) contains a noncentral Lie ideal of \( R \). This is used, among other things, to prove:

Theorem. Let \( R \) be a semiprime algebra over an infinite field, \( f_1, \ldots, f_t \) polynomials in disjoint sets of variables all noncentral for \( R \). Then, if \( R \) satisfies \( S_t[f_1, \ldots, f_t] \), \( R \) must satisfy \( S_t[x_1, \ldots, x_t] \).

Introduction. Let \( k \) be a commutative ring with 1. All rings will be unital associative \( k \)-algebras. Let \( k\{X\} = k\{x_1, x_2, x_3, \ldots\} \) be the free \( k \)-algebra in a denumerable number of (noncommuting) variables. We call a polynomial \( f(x_1, \ldots, x_n) \in k\{X\} \) central for a \( \mathcal{A} \)-algebra \( R \) if all specializations of \( f \) in \( R \) lie in the center of \( R \). Note that contrary to the common usage we do not require \( f \) to be nonvanishing on \( R \). The ring of \( n \times n \) matrices over \( R \) will be denoted by \( M_n(R) \) and we denote by \( e_{ij} \) \((1 \leq i, j \leq n)\) the standard matrix units in \( M_n(R) \). The standard polynomial of degree \( t \) will be denoted by \( S_t = S_t[y_1, \ldots, y_t] \).

In §1 we introduce the notion of the extended range of a polynomial \( f \) on \( R \), and we prove that this is a "large" subset of \( R \). Namely, if \( f \) is not central for \( R \) then the extended range is an invariant submodule which (under some mild assumptions on \( f \)) contains a noncentral Lie ideal of \( R \).

A polynomial \( f \) is central for \( R \) if \([f, y]\) is an identity for \( R \). One is tempted to ask: Does there exist a polynomial \( f \) which is not central but such that for some \( t > 2 \), \( S_t[f, y_1, \ldots, y_{t-1}] \) vanishes on \( R \)? It follows from [5] that the answer, at least in the manageable cases, is negative. In §2 we investigate a generalization of this problem. Let \( f_1, \ldots, f_t \) be polynomials in disjoint sets of variables. We say that the \( f_i \)'s are hypercommuting if none of them is central for \( R \) but \( S_t[f_1, \ldots, f_t] \) is an identity for \( R \). We prove that prime rings and semiprime algebras have no hypercommuting polynomials—except for some exceptional cases. We next specialize our results to the case \( f_i = x_i^n \) to prove a "higher commutativity theorem", that is, a commutativity theorem with commutators replaced by the standard polynomial.

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1. The extended range of a polynomial. Let $R$ be a $k$-algebra and $f(x_1, \ldots, x_n) \in k \{X\}$. We denote by $f[R]$ the (additive) $k$-submodule of $R$ generated by all the specializations of $f$ in $R$. We call $f[R]$ the extended range of $f$ on $R$.

We start with the following simple observation:

**Lemma 1.** Let $R$ be a $k$-algebra and $f \in k \{X\}$. Then,

(i) $f[R]$ is invariant under all homomorphisms of $R$ into $R$.

(ii) If $f$ is multilinear, $f[R]$ is invariant under all derivations of $R$, in particular, it is a Lie ideal of $R$.

**Proof.** The first part of the lemma is obvious. As for the second part, let $a_1, \ldots, a_n \in R$ and $a \to a'$ a $k$-derivation of $R$. It follows from the definition of a derivation that if $f$ is multilinear

$$f(a_1, \ldots, a_n) = \sum_{i=1}^n f(a_1, \ldots, a_i', \ldots, a_n) \in f[R].$$

Finally, a Lie ideal is an additive submodule invariant under all inner derivations.

We now would like to see how much of the second part of the lemma remains valid for a general polynomial $f$. To that purpose we examine the effect of multilinearization on the extended range $f[R]$. For details on the multilinearization process and the terms used the reader is referred to [4, pp. 15–19].

Recall that a polynomial $f$ is called blended in some variable $x_i$ if each monomial of $f$ has a positive degree in $x_i$. $f$ is called blended if it is blended in each variable appearing in it. Every polynomial may be written as a sum of blended polynomials. To be more precise: let $f = f(x_1, \ldots, x_n)$, for each subset $A \subseteq \{x_1, \ldots, x_n\}$ define $f_A$ to be the sum of the monomials in $f$ having positive degree exactly in the variables appearing in $A$. We have then $f = \sum_{A \subseteq \{x_1, \ldots, x_n\}} f_A$, the $f_A$'s are blended and uniquely determined by $f$. We call them the blended components of $f$. With the notation as above we have:

**Lemma 2.** $f[R] = \sum_A f_A[R]$.

**Proof.** $\subseteq$ is clear. For the converse we have to show that each $f_A[R]$ is contained in $f[R]$. Assume inductively that $f_B[R] \subseteq f[R]$ whenever $B \subset A$. Denote by $g$ the result of substituting 0 in $f$ for each variable not in $A$. Clearly $g = \sum_{B \subset A} f_B = \sum_{B \subset A} f_B + f_A$. By the definition of $g$, $g[R] \subseteq f[R]$ and so, since by our induction hypothesis the $f_B[R]$'s are included in $f[R]$, we conclude that $f_A[R] \subseteq f[R]$.

**Theorem 3.** Let $R$ be a $k$-algebra, $f(x_1, \ldots, x_n) \in k \{X\}$. $f[R]$ contains a nonzero (noncentral) Lie ideal of $R$—unless all the multilinearizations of all blended components of $f$ are identities (central) for $R$.

If $k$ is a field of characteristic zero and $f$ is not an identity (central) for $R$, then $f[R]$ contains a nonzero (central) Lie ideal.
Proof. Let $G$ be either $\{0\}$ or the center of $R$ as the case may be. By our assumption $f$ has a blended component $g$ with a multilinearization $h$ such that $h$ is not $G$ valued on $R$ (that is, $h[R] \not\subseteq G$). By Lemma 2, $g[R] \subseteq f[R]$ and clearly, since $h$ is a multilinearization of $g$, $h[R] \subseteq g[R]$. It now follows from part (ii) of Lemma 1 that $h[R]$ is a nonzero (noncentral) Lie ideal in $f[R]$.

Assume now $k$ is a field, $\chi(k) = 0$ and $f$ is not $G$ valued. By Lemma 2, $f$ must have a blended component $g$ which is not $G$ valued. As before, let $h$ be a multilinearization of $g$. It is well known that the assumption on the characteristic of $k$ implies $g[R] = h[R]$, and so $h[R]$ is a nonzero (central) Lie ideal in $f[R]$.

Remark. It is not difficult to see that $k$ does not really have to be of characteristic zero. It is enough that $\chi(k) = p > t$ where $t$ is the maximum of the degrees of $f$ in each variable.

In view of the theorem we shall say that a polynomial $f$ is strongly noncentral (strongly nonvanishing) on $R$ if it has a blended component with a multilinearization which is not central (vanishing) for $R$.

Our previous observations make it rather easy to classify all possible extended ranges for a matrix ring $R = M_n(k)$ over a field $k$. We first note that the only subspaces of $R$ invariant under all inner automorphisms are: 0, the scalar matrices, the trace zero matrices $[R, R]$, and the whole space $R$, except when $R = M_2(GF[2])$. This fact is not too difficult to prove directly, or one could apply theorems of Baxter [2, Theorem 2] and Sah [6, Proposition 2] which taken together imply the desired result. Either way, we deduce that if $f$ is a polynomial which is not central for $R = M_n(k)$, then $f[R] = R$ or $f[R] = [R, R]$—unless $R = M_2(GF[2])$. In order to handle this last exceptional case we need to assume that $f$ is strongly noncentral for $R$.

Assuming this, we know by Theorem 3 that $f[R]$ is not only an invariant subspace of $R$ but contains a noncentral Lie ideal of $R$. A direct computation will show that while $R = M_2(GF[2])$ does contain exceptional invariant subspaces and Lie ideals, the only invariant subspaces containing a noncentral Lie ideal are again $R$ and $[R, R]$. This proves:

**Corollary 4.** Let $R = M_n(k)$, $f \in k\{X\}$ a polynomial not central on $R$, then $f[R] = [R, R]$ or $f[R] = R$, except when $R = M_2(GF[2])$. The conclusion will still hold in the last case provided $f$ is strongly noncentral for $R$.

Remarks. 1. Most of the previous discussion can be done in a wider context. For that purpose one would use Amitsur's results about invariant subspaces of simple rings [1], rather than Baxter's.

2. It is easy to see that for $R = M_2(GF[2])$ the requirement that $f$ be strongly noncentral is indeed necessary. The extended range of $f(x) = x^5 + x^3$ on $M_2(GF[2])$ is the subspace of matrices of the form $(a+b \alpha \beta \alpha \beta)$. This happens since the multilinearization of $f$ (in characteristic 2) is $S_3$ which of course is an identity of $M_2(GF[2])$. 

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2. Hypercommuting polynomials. Let $R$ be a $k$-algebra, $f$ and $g$ two polynomials in $k\{X\}$. We shall say that $f$ dominates $g$ on $R$ if $f[R] \supseteq g[R]$.

**Lemma 5.** Let $R$ be a $k$-algebra, $T(x_1, \ldots, x_t)$ a multilinear polynomial in $k\{X\}$ (e.g. $S_t$), and $f_1, \ldots, f_t$, $g_1, \ldots, g_t$ arbitrary elements of $k\{X\}$ in disjoint sets of variables such that each $f_i$ dominates $g_i$ on $R$. Then $T(f_1, \ldots, f_t)$ dominates $T(g_1, \ldots, g_t)$ on $R$. In particular, if $T(f_1, \ldots, f_t)$ is an identity (central) for $R$, so is $T(g_1, \ldots, g_t)$.

**Proof.** It is easy to see from the multilinearity of $T$, that for each $i$, $T(g_1, \ldots, g_i, f_i, \ldots, f_t)$ dominates $T(g_1, \ldots, g_i, f_i, \ldots, f_t)$. The result now follows by the transitivity of dominance.

**Lemma 6.** For $n > 1$, $M_n(k)$ satisfies
\[ S_t[[x_1, x_2], [x_3, x_4], \ldots, [x_{2t-1}, x_{2t}]], \]
if and only if $t > 2n$ or $n = 2$, $t = 3$ and $3 = 0$ in $k$.

**Proof.** The fact that for $l > 2/7$, $M_l(k)$ satisfies
\[ S_t[[x_1, x_2], [x_3, x_4], \ldots, [x_{2l-1}, x_{2l}]], \]
follows from the Amitsur-Lewitski theorem. Since for $l < t$, $S_t$ can be expressed in terms of $S_l$, to prove the other direction it is enough to show that $M_n(k)$ does not satisfy
\[ S_{2n-1}[[x_1, x_2], [x_3, x_4], \ldots], \]
except in the one exceptional case. Let us first note that
\[ (1) \quad S_{2n-1}[e_{11}, e_{12}, e_{23}, \ldots, e_{n-1,n}, e_{nn-1}, e_{n-1,n-2}, \ldots, e_{21}] = 2 - e_{nn}. \]
Indeed, the only orders in which these matrix units yield nonvanishing products are the cyclic permutations of the given order. Among the $2n - 1$ possible products each $e_{ii}$ occurs twice except $e_{nn}$ which occurs only once. This, together with the fact that all these permutations are even, yields (1).

We similarly find that:
\[ (2) \quad S_{2n-1}[e_{nn}, e_{12}, e_{23}, \ldots, e_{n-1,n}, e_{nn-1}, \ldots, e_{21}] = (-1)^{n-1}(2 - e_{11}). \]
Subtracting (2) from (1) we get on the left a specialization of $S_{2n-1}$ in which all the arguments are commutators, while on the right we get $4 - e_{11} - e_{nn}$ when $n$ is even and $e_{11} - e_{nn}$ when $n$ is odd. These matrices do not vanish unless $n = 1$, or $n = 2$ and $3 = 0$ in $k$. To see that $2 \times 2$ matrices over a ring of characteristic 3 actually satisfy
\[ S_3[[x_1, x_2], [x_3, x_4], [x_5, x_6]], \]
note that $e_{22} - e_{11}, e_{12}$ and $e_{21}$ span $[R, R]$ for $R = M_2(k)$.

If $R$ is a prime PI ring we denote by $p.i. \deg R$ the least integer $n$ such that $R$ satisfies $S_{2n}$. The next theorem is well known and we include it for completeness.

**Theorem 7.** Let $R$ be a prime PI algebra over $k$, $p.i. \deg R = n$, $Z$ the
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Let \( R_0 = R \otimes_Z Q \) be the algebra of central quotients of \( R \). \( R_0 \) is central simple and \( n^2 \) dimensional over \( Q \) and satisfies the same identities as \( R \) [4, Theorem 2, p. 57]. If \( Q \) is a finite field then by the two Wedderburn theorems \( R_0 \cong M_n(Q) \) and we are done. If \( Q \) is infinite, let \( L \) be a splitting field for \( R_0 \); \( R_0 \otimes_Q L \cong M_n(L) \cong M_n(Q) \otimes_Q L \). Tensoring over infinite fields preserves identities [4, Lemma 1, p. 89] therefore \( R_0 \) and \( M_n(Q) \) have the same identities.

We can now prove our result about the nonexistence of hypercommuting polynomials for prime rings:

**Theorem 8.** Let \( R \) be a prime \( k \)-algebra, \( f_1, \ldots, f_r \) polynomials in disjoint sets of variables all noncentral for \( R \). If \( R \) satisfies \( S_1[f_1, \ldots, f_r] \) then \( R \) satisfies \( S_1[x_1, \ldots, x_n] \) except when \( x(R) = 3 \) and \( \text{p.i. \ deg \ } R = 2 \) or when \( R = M_n(GF[2]) \). In the last case the theorem still holds provided the \( f_i \)'s are strongly noncentral on \( R \).

**Proof.** Since the \( f_i \)'s are in disjoint sets of variables \( S_1[f_1, \ldots, f_r] \) is a nontrivial identity for \( R \) and \( R \) is a prime PI algebra. Let \( Z \) and \( Q \) be as in Theorem 7 and \( \text{p.i. \ deg \ } R = n > 1 \). By Theorem 7 we may assume \( R = M_n(Q) \). Note that \( M_n(Q) = M_2(GF[2]) \) if and only if \( R = M_2(GF[2]) \). Since none of the \( f_i \)'s is central for \( R \), by Corollary 4, \( f_i[R] \supseteq [R, R] \)---that is, \( f_i \) dominates \([x_{2i-1}, x_{2i}] \) on \( R \). By Lemma 5 this implies that \( S_i[f_1, \ldots, f_r] \) dominates \( S_i[\ldots, [x_{2i-1}, x_{2i}], \ldots] \) on \( R \) and therefore, by our assumption, the latter is an identity for \( R \). This implies, via Lemma 6, that \( S_i \) is an identity for \( R \)---unless \( n = 2 \) and \( x(R) = 3 \).

The following example indicates where we may run into trouble when trying to generalize Theorem 8 to semiprime rings.

**Example.** Let \( R = M_2(Z) \oplus M_3(Z_p) \), \( p \) a prime. Let \( f_1 \) be a (multilinear) polynomial which is an identity for \( M_2(Z) \) but is not central on \( M_3(Z_p) \) (e.g. \( f_1 = S_4(x_1, x_2, x_3, x_4) \)). Let \( f_2 \) be a (multilinear) polynomial which is an identity for \( M_3(Z_p) \) but not central for \( M_2(Z) \) (\( px_5 \) will do). \([f_1, f_2] \) is now an identity of \( R \) (as a matter of fact \( f_1 f_2 \) is already an identity), but neither \( f_1 \) nor \( f_2 \) are central for \( R \)--and of course \( R \) does not satisfy \( S_2[x_1, x_2] \).

We can however avoid these difficulties if we restrict ourselves to algebras over a field.

**Theorem 9.** Let \( R \) be a semiprime algebra over an infinite field \( k \) of characteristic \( \neq 3 \). Let \( f_1, \ldots, f_r \) be polynomials in disjoint sets of variables all noncentral for \( R \). If \( R \) satisfies \( S_1[f_1, \ldots, f_r] \) then \( R \) satisfies \( S_1[x_1, \ldots, x_n] \). If \( k \) is finite \( x(k) \neq 3 \) the theorem still holds provided the \( f_i \)'s are multilinear.

**Proof.** Assume \( R \) does not satisfy \( S_i \). Pick a prime \( P_0 \lhd R \) with \( \text{p.i. \ deg \ } R/P_0 \) maximal. By Theorem 8 some \( f_i \), say \( f_1 \), is central for \( R/P_0 \). Let \( P \lhd R \) be any prime, then \( \text{p.i. \ deg \ } R/P < \text{p.i. \ deg \ } R/P_0 \) and it follows easily
from the general theory (see [4]) that under our assumptions $f_i$ is central for $R/P$. Since $R$ is semiprime, $R$ is a subdirect sum of its prime homomorphic images, hence $f_i$ is central for $R$.

We now restrict the $f_i$'s in order to get a generalized commutativity theorem, namely:

**Theorem 10.** Let $R$ be a semiprime $k$-algebra satisfying $S[x_1^{n_1}, \ldots, x_t^{n_t}]$. Then $R$ satisfies $S[x_1, \ldots, x_t]$.

**Proof.** Assume first $R$ is prime and p.i. deg $R = n > 1$. Since $x_j^n$ is clearly not central on $R$, the result follows immediately from Theorem 8, except when $\chi(R) = 3$ and $n = 2$ or when $R = M_2(GF[2])$. To prove the theorem for these two exceptional cases it is enough to show that $2 \times 2$ matrices (over any commutative ring with $1 \neq 0$) do not satisfy an identity of the form $S_s[x_1^{n_1}, x_2^{n_2}]$ or $S_t[x_1^{n_1}, x_2^{n_2}, x_3^{n_3}]$. This is easily verified by the substitution $x_1 = e_{11} + e_{12}$, $x_2 = e_{11} + e_{21}$, $x_3 = 1$. If $R$ is semiprime choose again a prime $P_0 < R$ with p.i. deg $R/P_0$ maximal. By the previous discussion $R/P_0$ satisfies $S[x_1, \ldots, x_t]$, our choice of $P_0$ now implies that $R$ satisfies $S_r$.

Theorem 10 may naturally be considered as a commutativity theorem for higher degrees of commutativity. It would be interesting to see what can be said in this situation when the exponents are not fixed, i.e.:

**Question.** What can be said about a (say prime) ring $R$ satisfying

$$S_t[a_1^{n_1}, a_2^{n_2}, \ldots, a_t^{n_t}] = 0$$

where $n_i = n_i(a_1, a_2, \ldots, a_t)$?

This was only recently solved for the case $t = 2$ (see [3]). A general solution will certainly necessitate a different approach than the one employed here.

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**References**


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