

GENERALIZATIONS OF L'HÔPITAL'S RULE

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This paper is dedicated to Casper Goffman

ABSTRACT. An essential limit, similar to the concept of essential bounded functions, is defined and briefly discussed. Using the essential limit, l'Hôpital's rule is generalized to include the following theorem as a special case.
THEOREM. Let F, G be real-valued functions defined on the open interval (a, b) . Suppose that the approximate derivatives $F'_{\text{ap}}(x)$ and $G'_{\text{ap}}(x)$ exist finitely, $G'_{\text{ap}}(x) > 0$ for almost all x in (a, b) , and the extreme approximate derivatives of both F and G are finite nearly everywhere in (a, b) . Then $\lim_{x \rightarrow a^+} [F(x)/G(x)] = \text{ess } \lim_{x \rightarrow a^+} [F'_{\text{ap}}(x)/G'_{\text{ap}}(x)]$ provided that the essential limit in the right-hand side exists and that $\lim_{x \rightarrow a^+} F(x) = \lim_{x \rightarrow a^+} G(x) = 0$ or $\lim_{x \rightarrow a^+} G(x) = -\infty$.

1. Introduction. It is well known that l'Hôpital's rule is very useful in the evaluation of certain limits. The following is one version of this rule.

THEOREM 1. Let F, G be two real-valued functions defined on the open interval (a, b) . Suppose that the following conditions are satisfied:

- (H1) F and G are differentiable on (a, b) and $G'(x) \neq 0$ for all x in (a, b) ;
- (H2) $\lim_{x \rightarrow a^+} [F'(x)/G'(x)] = A$;
- (H3) $\lim_{x \rightarrow a^+} F(x) = 0 = \lim_{x \rightarrow a^+} G(x)$.

Then

$$(C) \lim_{x \rightarrow a^+} [F(x)/G(x)] = A.$$

Theorem 1 has been generalized in [3] by weakening the conditions involving the ordinary differentiability to the approximate Peano differentiability. In connection with the concept of the approximate limit, it is interesting to note that the theorem fails to hold true should the ordinary limit in (H2) and (C) be replaced by the approximate limit. In fact, in [7], functions F, G have been constructed to satisfy (H1) and (H3) and also that both $\text{ap } \lim_{x \rightarrow a^+} [F'(x)/G'(x)]$ and $\lim_{x \rightarrow a^+} [F(x)/G(x)]$ exist, but these two are *not* equal. Thus, in Theorem 1, the existence of the ordinary limit in (H2) cannot be weakened to that of the approximate limit even if one only wants to conclude that $\text{ap } \lim_{x \rightarrow a^+} [F(x)/G(x)] = A$. This leads us to the consideration of the essential limit (to be discussed in §2), which is a concept sandwiched in between that of the ordinary limit and that of the approximate limit. Using

Received by the editors February 22, 1977.

AMS (MOS) subject classifications (1970). Primary 26A03, 26A15; Secondary 26A24, 26A48.

Key words and phrases. Essential limit, ordinary limit, approximate limit, approximate Peano derivatives and derivates, generalized absolutely continuous functions, closed monotone functions, monotonicity theorem, l'Hôpital's rule.

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the essential limit, we are able to obtain some reasonable generalizations (§4) of Theorem 1 by replacing the condition (H2) by a weaker one. Meanwhile, the “everywhere” requirement in (H1) is also weakened to some “almost everywhere” requirements. This seems to be significant since many important functions are only differentiable almost everywhere.

The fundamental results used in the proof of our main results are the monotonicity theorems obtained recently in [4] and [5], which, together with the notations involved, will be briefly reviewed in §3.

2. Essential limits. Let a, b be two extended real numbers with $-\infty \leq a < b < +\infty$, and let F be a real-valued function defined at least almost everywhere on (a, b) . Then the *essential upper right limit of F at a* , denoted as $\text{ess lim sup}_{x \rightarrow a^+} F(x)$, is defined to be

$$\inf\{y: |\{x: F(x) > y\} \cap (a, c)| = 0 \text{ for some } c \in (a, b)\},$$

and similarly, the *essential lower right limit of F at a* , denoted as $\text{ess lim inf}_{x \rightarrow a^+} F(x)$, is defined to be

$$\sup\{y: |\{x: F(x) < y\} \cap (a, c)| = 0 \text{ for some } c \in (a, b)\},$$

where $|A|$ denotes the Lebesgue measure of A .

When these two (extreme one-sided) essential limits are equal, their common value, denoted as $\text{ess lim}_{x \rightarrow a^+} F(x)$, is called the *essential right limit of F at a* . Similarly, the essential left-sided and two-sided limits are defined in an obvious manner.

Recall that the ordinary and the approximate upper right limits can be characterized as follows (cf. Saks [9]):

$$\limsup_{x \rightarrow a^+} F(x) = \inf\{y: a \text{ is not an accumulation point of } \{x: x > a \text{ and } F(x) > y\}\},$$

$$\text{ap lim sup}_{x \rightarrow a^+} F(x) = \inf\{y: a \text{ is a point of dispersion for } \{x: x > a \text{ and } F(x) > y\}\}.$$

Hence, it follows immediately that we have the following inequalities:

$$\text{ap lim sup } F(x) \leq \text{ess lim sup } F(x) \leq \limsup F(x),$$

where all the limits are taken as $x \rightarrow a^+$, and similarly there are inequalities for lower limits. Then we conclude that the essential (approximate, resp.) limit exists and is equal to the ordinary (essential, resp.) whenever the latter exists. However, it is easy to see that the converses are not true. This is what is meant by the essential limit is a concept sandwiched in between those of the ordinary and the approximate limits.

3. Monotonicity theorems. In this section we fix the notations and quote two monotonicity theorems which will be used to state and to prove our main results in the next section. The following result is due to Lee [4].

THEOREM An ($n \geq 2$). Let F be a real-valued function defined on the compact interval I such that $F_{(n-1)}(x)$ exists finitely for every x in I . Suppose that $uF_{(n)}(x) \geq 0$ for almost all x in I and $uF_{(n)}(x) > -\infty$ for nearly all x in I , or, more generally, suppose only that $u_0F_{(n)}(x) \geq 0$ for almost all x in I and $u_0F_{(n)}(x) > -\infty$ for nearly all x in I . Then $F_{(n-1)}$ is monotone increasing on I .

Here, as in [4], for any positive integer k , $F_{(k)}(x)$ ($uF_{(k)}(x)$, $lF_{(k)}(x)$, resp.) is the k th (upper, lower, resp.) approximate Peano derivative (derivates, resp.) of F at x , and $u_0F_{(k)}(x)$ is the extended real number obtained by taking in the definition of $uF_{(k)}(x)$ the ordinary lim sup instead of the approximate lim sup. For convenience, we will use " $F_{(0)}(x)$ exists finitely" to mean that the function $F_{(0)}$ ($\equiv F$) is approximately continuous at x . Note then that we do not have Theorem An for $n = 1$ (see [4]).

The next theorem, also due to Lee [5], is a characterization for monotone functions.

THEOREM B. Let F be a real-valued function defined on the compact interval I . For the function F to be monotone increasing on I , it is necessary and sufficient that $\bar{F}(x) \geq 0$ for almost all x in I and that F is both uCM and [LACG] on I .

Here, just as in [5], $\bar{F}(x)$ denotes the upper derivate of F at x ; that F is uCM (*upper closed monotone*) on I means that F is monotone increasing on the closed interval $[\alpha, \beta]$ whenever F is so on the open interval $(\alpha, \beta) \subseteq I$; and that F is [LACG] (*generalized lower absolutely continuous*) on I means that I is the union of countably many closed sets on each of which F is lower absolutely continuous in the wide sense. A function F is said to be CM ([ACG], resp.) on I if both F and $-F$ are uCM ([LACG], resp.) on I .

Note that it is clear that every Darboux function and, hence, every approximately continuous function (cf. [10]) or every exact finite approximate Peano derivative (cf. [1]) is CM. Note also that an approximately continuous function F on the closed interval I is [LACG] on I provided that the lower approximate derivate of F is $> -\infty$ nearly everywhere in I (cf. [5]). For a discussion on when an exact ordinary (approximate, resp.) Peano derivative would be [ACG], we refer to [6], where some relations between Theorems An and B, as well as some of their applications have been briefly indicated.

We end this section with the following remark. Should the inequalities involving " ≥ 0 " in the sufficient conditions of both Theorems An and B be replaced by the ones with " > 0 ", then the "monotone increasing" in the conclusions can be replaced by "strictly increasing." This is so simply because a constant $F_{(n-1)}$ has $F_{(n)} = 0$ everywhere.

4. Generalizations of l'Hôpital's rule. Following the formulation of the ordinary l'Hôpital's rule in [8], we have the following generalization. Note that the proof here is essentially different from that in [8] in the sense that we

have to apply the monotonicity theorem directly rather than applying a generalized mean value theorem.

THEOREM I. *Let a, b be two extended real numbers with $-\infty < a < b < +\infty$, and let F, G be real-valued functions defined on the open interval (a, b) . Suppose that the following conditions are satisfied:*

(h1) $F_{(1)}(x)$ and $G_{(1)}(x)$ exist finitely and $G_{(1)}(x) > 0$ for almost all x in (a, b) ;

(h2) $\text{ess } \lim_{x \rightarrow a+} [F_{(1)}(x)/G_{(1)}(x)] = A$;

(h3)(i) $\lim_{x \rightarrow a+} F(x) = 0 = \lim_{x \rightarrow a+} G(x)$, or

(ii) $\lim_{x \rightarrow a+} G(x) = -\infty$;

(h4) $G, pG - F$ and $F - qG$ are [IACG] and uCM on every compact subinterval of (a, b) for all real numbers p in an open interval (A, p_0) provided that $A \neq +\infty$, and for all real numbers q in an open interval (q_0, A) provided that $A \neq -\infty$.

Then

(C) $\lim_{x \rightarrow a+} [F(x)/G(x)] = A$.

PROOF. First, note that it follows from Theorem B and the remark in the last section that the function G is strictly increasing on each compact subinterval of (a, b) , so that $G(\beta) - G(\alpha) > 0$ for all $\alpha, \beta \in (a, b)$ with $\alpha < \beta$.

Now, we consider the case that $-\infty < A < +\infty$. Let p be a number in (A, p_0) , and choose a number $p' \in (A, p)$. Then (h2) implies that there exists a real number $c \in (a, b)$ such that

$$F_{(1)}(x)/G_{(1)}(x) < p' \quad \text{for almost all } x \text{ in } (a, c),$$

and hence

$$F_{(1)}(x) < p' G_{(1)}(x) \quad \text{for almost all } x \text{ in } (a, c).$$

Then applying Theorem B to the function of $p'G - F$ on every compact interval $[\alpha, \beta] \subseteq (a, c)$, one obtains that

$$F(\beta) - F(\alpha) < p' [G(\beta) - G(\alpha)],$$

and hence

$$(1) \quad [F(\beta) - F(\alpha)]/[G(\beta) - G(\alpha)] < p' \quad \text{for all } [\alpha, \beta] \subseteq (a, c).$$

Suppose that (i) holds. Then letting $\alpha \rightarrow a+$ in (1), one has

$$(2) \quad F(\beta)/G(\beta) < p' \quad \text{for all } \beta \in (a, c).$$

Suppose that (ii) holds. Then fixing $\beta \in (a, c)$, there exists a $\delta \in (a, \beta)$ such that $G(x) < 0$ for all $x \in (a, \delta)$. Hence, multiplying (1) by the negative number $[G(\beta) - G(\alpha)]/G(\alpha)$, one obtains

$$(3) \quad F(\alpha)/G(\alpha) < p' - p' [G(\beta)/G(\alpha)] + [F(\beta)/G(\alpha)]$$

for all $\alpha \in (a, \delta)$. Letting $\beta \rightarrow a+$ in (2) or letting $\alpha \rightarrow a+$ in (3), one obtains in both cases that

$$\limsup_{x \rightarrow a^+} [F(x)/G(x)] \leq p' < p.$$

Since $p \in (A, p_0)$ is arbitrary, we conclude that

$$\limsup_{x \rightarrow a^+} [F(x)/G(x)] \leq A$$

whenever $-\infty \leq A < +\infty$.

Similarly, if $-\infty < A \leq +\infty$, one concludes that

$$\liminf_{x \rightarrow a^+} [F(x)/G(x)] \geq A.$$

From these, one concludes that

$$\lim_{x \rightarrow a^+} [F(x)/G(x)] = A.$$

The following Theorem II_n is an analogue of Theorem I. The proof, using Theorem AN instead of Theorem B, is similar to that of Theorem I and is omitted here.

THEOREM II_n ($n \geq 2$). *Let a, b be two extended real numbers with $-\infty \leq a < b \leq +\infty$, and let F, G be real-valued functions defined on the open interval (a, b) such that $F_{(n-1)}(x)$ and $G_{(n-1)}(x)$ exist finitely for all x in (a, b) . Suppose that the following conditions are satisfied:*

(h1. n) $F_{(n)}(x)$ and $G_{(n)}(x)$ exist finitely and $G_{(n)}(x) > 0$ for almost all x in (a, b) ;

(h2. n) $\text{ess } \lim_{x \rightarrow a^+} [F_{(n)}(x)/G_{(n)}(x)] = A$;

(h3. n) (i) $\lim_{x \rightarrow a^+} F_{(n-1)}(x) = 0 = \lim_{x \rightarrow a^+} G_{(n-1)}(x)$, or

(ii) $\lim_{x \rightarrow a^+} G_{(n-1)}(x) = -\infty$;

(h4. n) $u_0 G_{(n)}(x) > -\infty$, $u_0(pG - F)_{(n)}(x) > -\infty$ and $u_0(F - qG)_{(n)}(x) > -\infty$ for nearly all x in (a, b) and for all p in an open interval (A, p_0) provided that $A \neq +\infty$ and for all q in an open interval (q_0, A) provided that $A \neq -\infty$.
Then

$$(Cn) \lim_{x \rightarrow a^+} [F_{(n-1)}(x)/G_{(n-1)}(x)] = A.$$

We end this note by some discussions about Theorems I and II_n .

(A) Recalling our conventions made in §3, we see that the statement of Theorem II_n makes sense even for $n = 1$. However, we cannot prove Theorem II_n for $n = 1$ since Theorem AN fails to hold for $n = 1$. Nevertheless, should we replace condition (h4. n) by the following stronger one:

(h4. n)_s both the extreme n th approximate Peano derivatives of both F and G are finite nearly everywhere in (a, b) ,

then Theorem II_n holds even for $n = 1$. This is so since (h4,1)_s implies that both the approximate continuous functions $F_{(0)}$ ($\equiv F$) and $G_{(0)}$ ($\equiv G$) are [ACG] (cf. the second to last paragraph in §3) and, hence, so are their linear combinations, so that Theorem II_1 follows as a corollary of Theorem I. (Thus, we see that the theorem stated in the abstract is just a special case of Theorem I.) Whether (h4. n)_s (for $n \geq 2$) implies that $F_{(n-1)}$ and $G_{(n-1)}$ (assumed to exist everywhere) are [ACG] is an open question.

(B) Both Theorems I and II_n hold true should the ordinary limits in (h3),

(h3.n), (C) and (Cn) be replaced by the approximate limits.

(C) Both theorems hold true should the conditions $G_{(n)}(x) > 0$ and $\lim_{x \rightarrow a+} G_{(n-1)}(x) = -\infty$ be replaced by $G_{(n)}(x) < 0$ and $\lim_{x \rightarrow a+} G_{(n-1)}(x) = +\infty$. Should $G_{(n)}(x)$ exist finitely for every x in (a, b) , then " $G_{(n)}(x) > 0$ " can be replaced by " $G_{(n)} \neq 0$ " since an exact approximate Peano derivative has the Darboux property, so that " $G_{(n)} \neq 0$ " implies that "either $G_{(n)} > 0$ or $G_{(n)} < 0$."

(D) The analogous statements for left-sided and two-sided limits are, of course, also true.

(E) It also is clear from the proof that without assuming (h2) in Theorem I or (h2.n) in Theorem II_n, one can conclude that

$$\begin{aligned} \text{ess lim sup } [F_{(n)}(x)/G_{(n)}(x)] &\geq \lim \sup [F_{(n-1)}(x)/G_{(n-1)}(x)] \\ &\geq \lim \inf [F_{(n-1)}(x)/G_{(n-1)}(x)] \geq \text{ess lim inf } [F_{(n)}(x)/G_{(n)}(x)]. \end{aligned}$$

These generalize the following well-known ones (cf. [2]):

$$\lim \sup [F'/G'] \geq \lim \sup [F/G] \geq \lim \inf [F/G] \geq \lim \inf [F'/G'].$$

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