PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 66, Number 2, October 1977

ON POINTS AT WHICH A SET IS CONE-SHAPED

M. EDELSTEIN, L. KEENER AND R. O'BRIEN

ABSTRACT. A set S in a normed linear space X is said to be cone-shaped at $x \in X$ if there is a closed half-space that has x in its bounding hyperplane and contains $\{y \in S: [x, y] \subset S\}$. The point x is called a cone point. In this paper it is shown that if X has an equivalent uniformly convex and uniformly smooth norm and if S is a closed bounded subset with the finite visibility property for cone points (i.e., for every finite set F of cone points of S there is a point $z \in S$ such that $[z, y] \subset S$ for all $y \in F$), then S is starshaped.

Let S be a set in a normed linear space X and x a point in S. We say that S is cone-shaped at x if there is a closed half-space that contains $\mathrm{st}(x; S) \equiv \{y \in S$: the line segment joining x to y, $[x, y]$, is contained in $S\}$ and has x in its bounding hyperplane. The set $\mathrm{st}(x; S) = \mathrm{st}(x)$ is called the star of x (with respect to S) and x is called a cone point. Cone points have also been called regular points in the literature [12]. We favour the first term as being more descriptive. In [9] Krasnoselski proved that for S a closed subset of a finite dimensional space and $s \in S$, if $[0, s] \not\subset S$ then a cone point z of S exists with $0 \notin \overline{\mathrm{co}}(\mathrm{st}(z))$. (Here $\overline{\mathrm{co}}(A)$ denotes the closed convex hull of a set A.) This may be viewed as a type of separation property involving cone points. The question of whether a corresponding result is valid for infinite dimensional spaces is of special interest because of its implications regarding starshape (cf. [3], [7]); this question is also explicitly raised by Valentine [12, p. 84]. We prove here that Krasnoselski's result can be extended to infinite dimensional normed linear spaces X if either S is weakly compact, or, more interestingly, if S is norm closed and X has an equivalent norm which is both uniformly convex and uniformly smooth.

Before proving the main theorem, we show how Krasnoselski's result can be extended in the case that S is weakly compact.

PROPOSITION. *Let S be a weakly compact subset of a normed linear space X. Suppose that for some $s \in S$, $[0, s] \not\subset S$. Then a cone point z of S exists with* $0 \notin \overline{\mathrm{co}}(\mathrm{st}(z))$.

PROOF. Let E denote the completion of X and B the closure in E of the linear span of S. Then B is a weakly compactly generated Banach space, and

Received by the editors July 20, 1976 and, in revised form, April 7, 1977.
AMS (MOS) subject classifications (1970). Primary 52A30, 46B99.
Key words and phrases. Starshaped, superreflexive, finite visibility.

© American Mathematical Society 1977

so we may assume that the norm on B is smooth by a result of Amir and Lindenstrauss [2].

Since $[0, s] \not\subset S$, there exists a $\lambda \in (0, 1)$ and $\varepsilon > 0$ such that the open ball centered at λs of radius 2ε, $B(\lambda s, 2\varepsilon)$, is disjoint from S. By the weak compactness of S there exists a smallest $\lambda_0 > \lambda$ such that the closure of $B(\lambda_0 s, \varepsilon)$ intersects S. Let z belong to this intersection. Then if $C = \bigcup\{B(\lambda s, \varepsilon)\colon \lambda \in [0, \lambda_0]\}$, z belongs to the boundary of C and hence there exists a hyperplane H which supports C at z. Note that since z belongs to the boundary of $B(\lambda_0 s, \varepsilon) \subset C$, H also supports $B(\lambda_0 s, \varepsilon)$ at z. Now if H^+ denotes the closed half-space determined by H not containing C, we must have $\overline{\mathrm{co}}(\mathrm{st}(z)) \subset H^+$ and so $0 \notin \overline{\mathrm{co}}(\mathrm{st}(z))$. Indeed, if $[w, z] \subset S$ and $w \notin H^+$, a standard separation argument produces a second supporting hyperplane to $B(\lambda_0 s, \varepsilon)$ at z in contradiction to smoothness. $\square$

Our main result is the following:

THEOREM. *Let X be a normed linear space having an equivalent norm which is both uniformly convex and uniformly smooth, and suppose that $S \subset X$ is norm closed. If there is an $s \in S$ with $[0, s] \not\subset S$, then a cone point z of S exists such that $0 \notin \overline{\mathrm{co}}(\mathrm{st}(z))$.*

PROOF. Assume that the norm on X is uniformly convex and uniformly smooth. Choose $\varepsilon > 0$ such that for some $\lambda \in (0, 1)$, $B(\lambda s, \varepsilon) \cap S = \varnothing$. Let λ_0 be the greatest such λ. Then for any $\lambda > \lambda_0$, $B(\lambda s, \varepsilon) \cap S \neq \varnothing$. By uniform convexity, the set of points in X that have nearest points in S is dense in X [6], [11]. Hence, for any fixed $\delta > 0$ and $\lambda > \lambda_0$, we can find an x with $\|x - \lambda s\| < \delta$ and such that x has a nearest point z in S. By choosing δ small enough, we can also assume $\|\lambda s - z\| < \varepsilon$. Now if H denotes the hyperplane of support to $B(x, \|x - z\|)$ at z, by the same smoothness argument as used in the proposition, we must have $\overline{\mathrm{co}}(\mathrm{st}(z)) \subset H^+$, where H^+ denotes the closed half-space determined by H not containing x. To complete the proof we show that for some λ close to λ_0 and δ small, $0 \notin H^+$.

For any nonzero $y \in X$, let f_y denote the functional of norm one which supports the ball $B(0, \|y\|)$ at y. Then the hyperplane of support to $B(x, \|x - z\|)$ at z is given by

$$H = x + f_{z-x}^{-1}(\|z - x\|).$$

Since $\|\lambda s - z\| < \varepsilon$ implies $\|\lambda_0 s - (z - (\lambda - \lambda_0)s)\| < \varepsilon$, there exists a $\lambda_1 \in (0, \lambda - \lambda_0)$ such that $w = z - \lambda_1 s$ belongs to the boundary of $B(\lambda_0 s, \varepsilon)$. Let $H_1 = \lambda_0 s + f_{w-\lambda_0 s}^{-1}(\varepsilon)$ denote the hyperplane of support to $B(\lambda_0 s, \varepsilon)$ at w. Then for some $\lambda' > \lambda_0$, $\lambda' s \in H_1$ since $z - (\lambda - \lambda_0)s \in B(\lambda_0 s, \varepsilon)$. Hence $f_{w-\lambda_0 s}(\lambda' s - \lambda_0 s) = \varepsilon$ or since $\lambda' > \lambda_0$,

$$f_{w-\lambda_0 s}(\lambda_0 s) > 0.$$

On the other hand, if H intersects the segment $[0, \lambda_0 s]$, then $f_{z-x}(0) = 0 \geqslant \|z - x\| + f_{z-x}(x)$ or

$$f_{z-x}(x) \leqslant -\|z - x\|.$$

Uniform smoothness and (1) now imply that for λ close to λ_0 and δ small, (2) cannot occur. Hence for z corresponding to such an H, $0 \notin \overline{\mathrm{co}}(\mathrm{st}(z))$. $\square$

Since the Krasnoselski result was an essential tool in characterizing closed, bounded, starshaped sets in finite dimensional spaces (see e.g. [12]), our extension can be similarly used in the more general setting. Recall that a set S is called starshaped if there is a point $x \in S$ such that $\mathrm{st}(x) = S$. A set S is said to have the finite visibility property for cone points if for every finite set F of cone points of S, there is a point $x \in S$ such that $F \subset \mathrm{st}(x)$. Using this last concept we may state the following:

COROLLARY. *Suppose that either*

(i) *S is a weakly compact subset of a normed linear space X, or*

(ii) *S is a closed, bounded subset of a normed linear space X which has an equivalent uniformly convex and uniformly smooth norm.*

Then S having the finite visibility property for cone points implies that S is starshaped.

PROOF. By either the proposition (case (i)) or the theorem (case (ii)) the set of cone points of S, $K(S)$, is nonempty. Combining the finite visibility hypothesis with weak compactness, we see that there exists $y \in \bigcap \{\overline{\mathrm{co}}(\mathrm{st}(x)): x \in K(S)\}$. By translating y to 0 and applying either the proposition or the theorem, we see that $\mathrm{st}(y) = S$. $\square$

REMARKS. 1. Work by R. C. James and P. Enflo on superreflexive spaces and a procedure for averaging norms due to E. Asplund can be used to show that the Banach spaces which have an equivalent norm which is both uniformly convex and uniformly smooth are exactly the superreflexive Banach spaces (see e.g. [4] or [5]).

2. The relationship between starshape and finite visibility (for all finite subsets) of weakly closed and bounded sets in infinite dimensional spaces was first studied in [7]. In [1] these results were extended to norm closed sets in a Hilbert space and in [3] to norm closed sets in any reflexive Banach space. The above corollary improves the previously obtained results (within the class of superreflexive spaces) in that finite visibility is needed for cone points only. It is not known whether the corollary holds in an arbitrary reflexive Banach space.

3. An interesting question presents itself. Is it possible to find a closed, bounded set in a Banach space, that has no cone points? There are closed, bounded, convex sets in incomplete normed spaces with no support points and hence no cone points [8], and there is an example [10] of a closed, bounded, convex set in a Fréchet space, which has no support points.

The authors are grateful to the referee for suggestions which contributed to a clearer presentation of the main result.

This research was supported by the National Research Council of Canada, Grants A-3999 and A-8755. Research by the last-named author was supported by an I. W. Killam Postdoctoral Fellowship from Dalhousie University.

REFERENCES

1. R. Alexander and M. Edelstein, *Finite visibility and starshape in Hilbert space* (preprint).

2. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. (2) **88** (1968), 35–46.

3. J. Borwein, M. Edelstein and R. O'Brien, *Visibility and starshape*, J. London Math. Soc. (2) **14** (1976), 313–318.

4. M. M. Day, *Normed linear spaces*, third ed., Springer-Verlag, Berlin and New York, 1973.

5. J. Diestel, *Geometry of Banach spaces–selected topics*, Springer-Verlag, Berlin and New York, 1975.

6. M. Edelstein, *On nearest points in uniformly convex Banach spaces*, J. London Math. Soc. **43** (1967), 375–377.

7. M. Edelstein and L. Keener, *Characterizations of infinite-dimensional and nonreflexive spaces*, Pacific J. Math. **57** (1975), 365–369.

8. V. Klee, *Extremal structure of convex sets*. II, Math. Z. **69** (1958), 90–104.

9. M. Krasnoselski, *Sur un critère pour qu'un domaine soit étoelée*, Mat. Sb. **19** (1946), 309–310.

10. N. T. Peck, *Support points in locally convex spaces*, Duke Math. J. **38** (1971), 271–278.

11. S. B. Stečkin, *Approximation properties of sets in normed linear spaces*, Rev. Math. Pures Appl. **8** (1963), 5–18. (Russian)

12. F. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX (B3H 3J5), NOVA SCOTIA, CANADA