A NOTE ON GENERATORS OF SEMIGROUPS

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Abstract. The generator $T$ of a norm-continuous semigroup of identity preserving positive linear mappings on a C*-algebra $A$ is characterized as one that satisfies $T(u^*u) > u^*T(u) + T(u*)u$ for all unitary elements $u$ in $A$.

1. Introduction. Since 1948 the analytical theory of semigroups has made vigorous progress. In that year K. Yosida and E. Hille (see [2, p. 363]) independently characterized the infinitesimal generators for strongly continuous semigroups of contractions on a normed linear space as those densely defined closed linear operators $T$ with $\|(\lambda I - T)^{-1}\| < \lambda^{-1}$ for all $\lambda > 0$. Later in 1960 G. Lumer and R. S. Phillips [5] found another description of those generators as those densely defined “dissipative” operators $T$ with range $[I - T] = X$, where $X$ is a semi-inner-product space. If the semigroup of contractions $\alpha(t)$ is norm-continuous, its infinitesimal generator becomes just bounded dissipative. In this note we are concerned with the norm-continuous semigroups of identity preserving positive linear mappings on a C*-algebra and find a new characterization of their infinitesimal generators in terms of the C*-algebra ordering. It would be more desirable to have a description of the generators of strongly continuous semigroups of positive (or completely positive) linear mappings, and that is currently being investigated by this author.

Lindblad’s recent work [3] on generators of one-parameter semigroups of completely positive identity preserving linear mappings on a C*-algebra showed an analogous result. As it is known that “positive” case does not follow automatically from “completely positive” case, one part of the proof of Theorem 1 is the same as that used in [3] and the other part is quite different.

2. Preliminary. Let $X$ be a Banach space, $X^*$ its dual space and $B(X)$ the Banach algebra of all bounded linear operators (or mappings) on $X$. Given a linear functional $f$ on $X$ and an element $x$ in $X$, we define a linear functional $(x, f)$ on $B(X)$ by $(x, f)(T) = f(T(x))$ for $T$ in $B(X)$. It is easily seen that (i) $\|(x, f)\| < \|x\| \|f\|$, (ii) $(\alpha x, f) = \alpha (x, f)$ for scalar $\alpha$, (iii) $(x_1 + x_2, f) =$
The spatial numerical range \( V(T) \) of a given linear mapping \( T \) in \( B(X) \) is defined as
\[
V(T) = \{ (x,f)(T) | f(x) = 1 = \|f\| = \|x\| \}.
\]

Let \( B \) be a unital Banach algebra. Given \( f \) in \( B^* \) and \( x \) in \( B \), we define a linear functional on \( B \) as \( f_x(y) = f(yx) \) for all \( y \) in \( B \) and observe (i) \( \|f_x\| \leq \|f\| \|x\| \), (ii) \( f_{ax} = af_x \) for scalar \( a \), (iii) \( f_{x_1 + x_2} = f_{x_1} + f_{x_2} \).

The numerical range \( V(T, B) \) of a given element \( T \) in \( B \) is defined as
\[
V(T, B) = \{ f(T) | f(I) = 1 = \|f\| = \|x\| \},
\]
where \( I \) is the identity element in \( B \). It is known that \( V(T, B) = \{ f_x(T) | f(x) = 1 = \|f\| = \|x\| \} \). In the following we list several properties needed in this note whose proofs can be found in [1] (see Theorem 4, p. 84, Theorem 4, p. 30, Theorem 4, p. 28 and Theorem 6, p. 30) and [4].

**Proposition 1.** \( \overline{\text{co}} V(T) = V(T, B) \), where \( \overline{\text{co}} V(T) \) is the closed convex hull of \( V(T) \). As a consequence \( V(T, B) \) is always a closed convex subset in the complex plane \( C \).

**Proposition 2.**
\[
\text{Max}\{ \Re \alpha; \alpha \in V(T, B) \} = \inf_{\|a\| > 0} \frac{1}{\alpha} \left\{ \|I + \alpha T\| - 1 \right\} = \lim_{\alpha \to 0^+} \frac{1}{\alpha} \left\{ \|I + \alpha T\| - 1 \right\}.
\]

**Proposition 3.**
\[
\text{Max}\{ \Re \alpha; \alpha \in V(T, B) \} = \sup_{\|a\| > 0} \frac{1}{\alpha} \left\{ \log \|\exp(\alpha T)\| \right\} = \lim_{\alpha \to 0^+} \frac{1}{\alpha} \left\{ \log \|\exp(\alpha T)\| \right\}.
\]

**Proposition 4.** Let \( x \) be in \( B \). Then \( \Re \lambda < 0 \) for all \( \lambda \) in \( V(x, B) \) if and only if \( \|\exp(tx)\| < 1 \) for all \( t > 0 \).

**Definition.** An element \( x \) in \( B \) is said to be dissipative if \( \Re \lambda < 0 \) for all \( \lambda \) in \( V(x, B) \).

A linear mapping \( T \) of a \( C^* \)-algebra \( \mathfrak{A} \) into itself is called selfadjoint if \( T(x^*) = T(x)^* \) for all \( x \) in \( \mathfrak{A} \), and is called positive if \( T(x) \) is positive for all positive elements \( x \) in \( \mathfrak{A} \). It is easy to see that a positive linear mapping is also selfadjoint.

3. **The main theorem.** Let \( e^{itT} \) be a one-parameter norm-continuous semigroup of positive linear mappings on a \( C^* \)-algebra \( \mathfrak{A} \) with \( e^{iT}(1) = 1 \), and \( T \) its generator (1 is the identity element in \( \mathfrak{A} \), and all \( C^* \)-algebras considered in this note have identity element). From \( T(x) = \lim_{t \to 0^+} (e^{it}(x) - x)/t \) \( x \in \mathfrak{A} \), we see that \( T \) is selfadjoint and \( T(1) = 0 \). In the following is
the theorem in which the generator $T$ is characterized by $C^*$-algebra ordering structure.

**Theorem 1.** Let $T$ be a bounded selfadjoint linear mapping on a $C^*$-algebra $\mathcal{A}$ with $T(1) = 0$. The following two conditions are equivalent:

(i) $T$ is dissipative.

(ii) $T(u^*)u + u^* T(u) < 0$ for all unitary elements $u$ in $\mathcal{A}$.

**Proof.** (ii) $\Rightarrow$ (i). The argument used here is the same as that in Proposition 4 in [3]. For the sake of completeness we do it as below. Because of Proposition 2 and a theorem due to Russo and Dye [6], it suffices to show that

$$r(T) = \lim_{t \to 0^+} \sup_{u} \frac{1}{t} \left\{ \|u + tT(u)\| - 1 \right\} < 0,$$

where the supremum is taken over all unitary elements in $\mathcal{A}$.

$$\|u + tT(u)\|^2 = \|u^* u + u^* tT(u) + tT(u^*)u + t^2 T(u^*)T(u)\|$$

$$< \|1 + t^2 T(u^*)T(u)\| < 1 + t^2 \|T\|^2.$$  

Hence, $\|I + tT\|^2 < 1 + t^2 \|T\|^2$, and

$$((\|I + tT\| - 1)(\|I + tT\| + 1) < t^2 \|T\|^2, $$

hence,

$$((\|I + tT\| - 1)/t < t\|T\|^2/(\|I + tT\| + 1).$$

Therefore $r(T) < 0$.

(i) $\Rightarrow$ (ii). By a theorem due to Russo and Dye [6] we have

$$\|I + tT\|^2 = \sup_{u: \text{unitary in } \mathcal{A}} \|(I + tT)(u)\|^2$$

$$= \sup_{u} \|(u + tT(u))^*(u + tT(u))\|.$$ 

Hence, for any state $\phi$ on $\mathcal{A}$ and unitary element $u$ in $\mathcal{A}$, we have

$$\phi((u + tT(u))^*(u + tT(u))) < \|I + tT\|^2,$$

$$\phi(u^* u + t(T(u^*)u + u^* T(u)) + t^2T(u^*)T(u)) < \|I + tT\|^2,$$

$$t\phi(T(u^*)u + u^* T(u)) + t^2\phi(T(u^*)T(u)) < \|I + tT\|^2 - 1,$$

$$\phi(T(u^*)u + u^* T(u)) + t\phi(T(u^*)T(u))$$

$$< ((\|I + tT\| - 1)/t)(\|I + tT\| + 1).$$

Taking the limit for both sides as $t \to 0^+$ we have

$$\phi(T(u^*)u + u^* T(u)) < r(T) \cdot 2 < 0$$

because $r(T) < 0$. Therefore $T(u^*)u + u^* T(u) < 0$. Q.E.D.

**Corollary 1.** The generator $T$ of a norm-continuous semigroup of identity preserving positive linear mappings on a $C^*$-algebra satisfies condition (ii) in Theorem 1.
PROOF. It is because of Proposition 4 and Theorem 1. Q.E.D.

Without the assumption $T(1) = 0$ in the above corollary, condition (ii) can be replaced by

(ii)' $T(1) + u^*T(1)u - T(u^*)u - u^*T(u) > 0$, for all unitary $u$ in $\mathcal{A}$.

Define $T'(x) = T(x) - \frac{1}{2}(T(1)x + xT(1))$ for all $x$ in $\mathcal{A}$. Since

$$T'(u^*)u + u^*T'(u) = T(u^*)u + u^*T(u) - u^*T(1)u - T(1),$$

condition (ii) holds for $T'$ if and only if condition (ii)' holds for $T$. Denote $x \rightarrow Kx + xK$ by $\{K, x\} = T_K(x)$. The semigroup generated by $T_K$ is $\exp iT_K(x) = e^{itK}xe^{itK}$. The semigroup generated by $T' + T'' (= T)$, where $T'' = T_{(1/2)T(1)}$, is given by the Lie-Trotter formula

$$\exp t(T' + T'') = \lim_{n \rightarrow \infty} \left[ \exp\left(\frac{tT'}{n}\right)\exp\left(\frac{tT''}{n}\right) \right]^n.$$ 

Hence $\exp t(T' + T'')$ is positive if $T$ satisfies (ii)'. Conversely, suppose that $\exp iT$ is positive. Then $T$ is selfadjoint. By the Lie-Trotter formula the semigroup generated by $T - T'' (= T')$ is positive, and $(\exp tT')(1) = 1$ for all $t > 0$. By Corollary 1, condition (ii) holds for $T'$. Therefore condition (ii)' holds for $T$. We have concluded

**Corollary 2.** Let $T$ be a bounded selfadjoint linear mapping from $\mathcal{A}$ into itself. Then $\exp iT$ is positive for all $t > 0$ iff condition (ii)' holds for $T$.

**References**


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