ON THE VON NEUMANN ALGEBRA OF
AN ERGODIC GROUP ACTION

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ABSTRACT. We give a criterion that an ergodic action be amenable in terms of the operator algebra associated to it by the Murray-von Neumann construction.

The notion of an amenable ergodic action of a locally compact group was introduced by the author in [7]. (The reader is cautioned that this is not the same as the notion introduced by Greenleaf in [3]; see [7] for a discussion of this point.) In this note, we give a criterion for an action of a countable discrete group to be amenable in terms of the von Neumann algebra associated to it by the classical Murray-von Neumann construction.

We recall that J. T. Schwartz has introduced the following property of von Neumann algebras [6, p. 168]. If $A$ is a von Neumann algebra on a Hilbert space $H$, $A$ is said to have property P if for any $T \in B(H)$, the closed (weak operator topology) convex hull of \( \{ U^*TU \mid U \in U(A) \} \) contains an element of $A'$, where $U(A)$ is the unitary group of $A$, and $A'$ is, as usual, the commutant of $A$. If $G$ is a countable discrete group and $R(G)$ the von Neumann algebra generated by the right regular representation of $G$, then $R(G)$ has property P if and only if $G$ is amenable [5, Proposition 4.4.21]. More generally, Schwartz shows that if $G$ is amenable and acts ergodically on a Lebesgue space $(S, \mu)$, then $R(S \times G)$, the algebra of the Murray-von Neumann construction (described below) has property P [6, p. 198]. Conversely, if $R(S \times G)$ has property P and $\mu$ is finite and $G$-invariant, then $G$ must be amenable [6, p. 200]. The point of this note is to prove a stronger converse, without the assumption of a finite invariant measure. Namely, we show that if $R(S \times G)$ has property P, then $S$ is an amenable $G$-space (definition below). The result quoted above in the case of finite invariant measure then follows from [7, Proposition 4.4], which asserts that if $S$ is an amenable $G$-space with finite invariant measure (or mean), then $G$ itself must be amenable.

We recall the Murray-von Neumann construction. Let $S$ be a standard Borel space, $G$ a countable discrete group with a right Borel action of $G$ on $S$. We suppose $\mu$ is a probability measure on $S$ quasi-invariant and ergodic.
We let \( r(s, g) \) be the Radon-Nikodym cocycle of the action, i.e. a Borel function such that 
\[
du(\sigma g) = r(s, g)\du(s).
\]
Let \( U_g : L^2(G) \to L^2(G) \) be the right regular representation of \( G \), and define a unitary representation \( \tilde{U}_g : L^2(S \times G) \to L^2(S \times G) \) by
\[
(\tilde{U}_g f)(s, h) = f(sg, hg)r(s, g)^{1/2}.
\]
Here \( S \times G \) has the product measure of \( \mu \) with Haar measure. Let \( V_g \) be the left regular representation of \( G \) on \( L^2(G) \) and define \( \tilde{V}_g : L^2(S \times G) \to L^2(S \times G) \) by
\[
(\tilde{V}_g f)(s, h) = f(s, g^{-1}h).
\]
If \( f \in L^\infty(S) \), then \( f \) defines a multiplication operator \( M_f \) on \( L^2(S \times G) \) by 
\[
(M_f h)(s, g) = f(s)h(s, g)
\]
and a multiplication operator \( N_f \) by 
\[
(N_f h)(s, g) = f(\sigma g)h(s, g).
\]
We let \( L \) be the von Neumann algebra generated by \( \{ V_g, N_f \} \) and \( R \) the von Neumann algebra generated by \( \{ \tilde{U}_g, N_f \} \). Then \( R' = L \), and there is a unitary involution \( J \) on \( L^2(S \times G) \) such that \( JRJ = L \) [1, pp. 137–138]. Thus \( R \) and \( L \) are spatially isomorphic and one will have property \( P \) if and only if the other does.

The definition of an amenable ergodic action given in [7] is based upon an “invariant section property” and is motivated by the virtual subgroup viewpoint of Mackey. We review the definition. Suppose \( E \) is a separable Banach space and \( \gamma : S \times G \to \text{Iso}(E) \) is a Borel cocycle, where \( \text{Iso}(E) \) is the group of isometric isomorphisms of \( E \) with the Borel structure of the strong operator topology. Let \( E_1^* \) be the unit ball in the dual of \( E \) with the \( \sigma(E^*, E) \) topology. Then there is an induced adjoint cocycle \( \gamma^* : S \times G \to \text{Homeo}(E_1^*) \), 
\[
\gamma^*(s, g) = (\gamma(s, g)^*)^{-1}.
\]
A Borel function \( \phi : S \to E_1^* \) is called an invariant section for \( \gamma \) if for each \( g \in G \), \( \gamma^*(s, g)\phi(\sigma g) = \phi(s) \) for almost all \( s \in S \). Now suppose that for each \( s \), \( A_s \subset E_1^* \) is a compact convex set, that \( \{(s, x) : x \in A_s\} \) is a Borel subset of \( S \times E_1^* \), and that for each \( g \), \( \gamma^*(s, g)A_{\sigma g} = A_s \) for almost all \( s \). Then \( S \) is called an amenable \( G \)-space if for each such cocycle \( \gamma \) and each such collection \( \{ A_s \} \), there is an invariant section \( \phi \) with \( \phi(s) \in A_s \) for almost all \( s \). In [7] it is shown that any ergodic action of any amenable group is amenable, but that nonamenable groups can also have amenable actions. For example the range-closure of a cocycle [4] of an amenable action into any locally compact group is amenable [7, Theorem 3.3].

**Theorem.** If \( G \) is a countable discrete group acting ergodically on \((S, \mu)\) and the von Neumann algebra \( R \) (or equivalently, \( L \)) has property \( P \), then \( S \) is an amenable \( G \)-space. (Here \( \mu \) is a quasi-invariant probability measure.)

**Proof.** If \( f \in L^2(S \times G) \), define 
\[
f_s(g) = f(sg, g)r(s, g)^{1/2}.
\]
Then \( f_s \in L^2(G) \) for almost all \( s \), and a straightforward calculation shows that \( f \to f^\otimes f \) is a unitary isomorphism of \( L^2(S \times G) \). One further readily verifies that under this isomorphism, \( \tilde{U}_g \) corresponds to \( f^\otimes U_g \) and that if \( T \in B(L^2(S \times G)) \) corresponds to the decomposable operator \( f^\otimes T_s \) in...
B(\mathcal{L}_2(G))$, then $T$ commutes with $\hat{V}_g$ if and only if for each $g$, $V^{-1}_g T_s V_g = T_{sg}$ for almost all $s$. Since the operators $\hat{U}_g$ and $M_f, f \in L^\infty(S)$, correspond to decomposable operators, it follows that every element of $R$ is decomposable with respect to this direct integral decomposition of $L^2(S \times G)$.

Now suppose that $L$ has property $P$. Then [5, Proposition 4.4.15] there is a linear mapping $P: B(L^2(S \times G)) \to R (= L'^*)$ such that

(i) $\|P\| < 1, P(I) = I, T > 0$ implies $P(T) > 0$.
(ii) $P(T) \in C(T)$, where $C(T)$ is the closed convex hull of $\{UTU^*| U \in U(L)\}$.

(iii) $P(S_1TS_2) = S_1P(T)S_2$ if $S_1, S_2 \in R$.

If $f \in L^\infty(S \times G)$, we have the multiplication operator $M_f \in B(L^2(S \times G))$, and since each element of $R$ is decomposable, we can write $P(M_f) = \int f \hat{M}_f d\mu$. For $f \in L^\infty(S \times G)$, write $(f \cdot g)(s, h) = f(sg, hg)$. Then $\hat{U}_g P(M_f)\hat{U}_g^{-1} = P(\hat{U}_g M_f \hat{U}_g^{-1}) = P(M_{f,g})$. Thus $\hat{U}_g (f \hat{M}_f)\hat{U}_g^{-1} = f \hat{M}_{f,g}$, i.e. $\int f \hat{U}_g T_s \hat{U}_g^{-1} = f \hat{T}_s$ for almost all $s$. We also note that since $f \hat{M}_f \in L'^*$, it follows from the remarks in the preceding paragraph that for each $g$, $V^{-1}_g T_s V_g = T_{sg}$ a.e.

For $f \in L^\infty(S \times G)$, define $\sigma(f)(s) = \langle T_s^g \chi_e | \chi_e \rangle$ where $\chi_e \in L^2(G)$ is the characteristic function of the identity. Then $\sigma: L^\infty(S \times G) \to L^\infty(S)$ and this map has the following properties:

(i) $\|\sigma(f)\|_\infty \leq \|f\|_\infty$.
(ii) $\sigma(1) = 1$.
(iii) If $f > 0, \sigma(f) > 0$.
(iv) If $A \subset S$ is measurable, $\sigma(f \chi_A \chi_G) = \sigma(f) \chi_A$.
(v) $\sigma(f \cdot g) = \sigma(f) \cdot g$.

Properties (i)-(iv) follow in a straightforward manner from the properties of the map $P$ listed above and some elementary properties of direct integrals of operators. To see (v), note that for each $g$ and almost all $s$,

$$\sigma(f \cdot g)(s) = \langle T_s^g \chi_e | \chi_e \rangle = \langle U_g T_s^g U_g^{-1} \chi_e | \chi_e \rangle$$
$$= \langle U_g V_g T_s V_g^{-1} U_g^{-1} \chi_e | \chi_e \rangle = \langle T_{sg} V_g^{-1} U_g^{-1} \chi_e | V_g^{-1} U_g^{-1} \chi_e \rangle$$
$$= \langle T_s^g \chi_e | \chi_e \rangle = \sigma(f)(sg).$$

We now demonstrate how the map $\sigma$ can be used to show that $S$ is amenable. Suppose $E, \gamma,$ and $(A_j)$ are as in the discussion preceding the statement of the theorem. Since $(s, A_s)$ is Borel, there is a measurable function $b: S \to E^*_\gamma$ such that $b(s) \in A_s$ for almost all $s \in S$. Define $F: S \times G \to E^*_\gamma$ by $F(s, g) = \gamma^*(s, g^{-1})b(sg^{-1})$. Then for each $\theta \in E, (s, g) \to \langle \theta, F(s, g) \rangle$ is in $L^\infty(S \times G)$ (where $\langle , \rangle$ is the duality of $E$ and $E^*$), and the map $E \to L^\infty(S), \theta \to \sigma(\langle \theta, F(s, g) \rangle)$ is linear with norm $\leq 1$. It follows from [2, p. 582] that there is a measurable function $a: S \to E^*_\gamma$ such that $\sigma(\langle \theta, F(s, g) \rangle)(s) = \langle \theta, a(s) \rangle$ a.e. We claim that $a(s)$ is the required invariant section.

**Lemma.** For all essentially bounded and measurable $\theta: S \to E$,
\[ \sigma(\langle \theta(s), F(s, g) \rangle)(s) = \langle \theta(s), a(s) \rangle \quad \text{a.e.} \]

**Proof.** Suppose first that \( \theta \) is a simple function, i.e. \( \theta(s) = \sum_{i} \theta_i \chi_{A_i}(s) \) where \( \{A_i\} \) is a countable partition of \( S \) and \( \theta_i \in E \). Fix \( j \). Then for almost all \( s \in A_j \), by property (iv) of \( \sigma \),
\[ \sigma(\langle \theta(s), F(s, g) \rangle)(s) = \sigma(\langle \theta(s), F(s, g) \rangle \chi_{A_j \times G}) = \sigma(\langle \theta_j, F(s, g) \rangle \chi_{A_j \times G}) \]
\[ = \sigma(\langle \theta_j, F(s, g) \rangle)(s) = \langle \theta_j, a(s) \rangle = \langle \theta(s), a(s) \rangle. \]
Since \( j \) is arbitrary, this lemma holds for simple \( \theta \). If \( \theta \) is arbitrary, then there are simple functions \( \theta_n \) with \( \|\theta_n - \theta\|_{\infty} \to 0 \) by virtue of the separability of \( E \).
Since \( F(s, g) \in E^* \), \( \langle \theta_n(s), F(s, g) \rangle \to \langle \theta(s), F(s, g) \rangle \) in \( \| \| \infty \) on \( S \times G \), and by the norm continuity of \( \sigma \),
\[ \sigma(\langle \theta_n(s), F(s, g) \rangle) \to \sigma(\langle \theta(s), F(s, g) \rangle) \]
in \( L^\infty(S) \). Clearly \( \langle \theta_n(s), a(s) \rangle \to \langle \theta(s), a(s) \rangle \) a.e., and the lemma follows.

**Corollary.** Suppose \( \alpha : S \to \text{Iso}(E) \) is measurable. Then for all \( \theta \in E \),
\[ \sigma(\langle \theta, \alpha(s)^* F(s, g) \rangle) = \langle \theta, \alpha(s)^* a(s) \rangle \]

**Proof.** This is equivalent to \( \sigma(\langle \alpha(s) \theta, F(s, g) \rangle) = \langle \alpha(s) \theta, a(s) \rangle \) which holds by the lemma.

We now show that \( a(s) \) is an invariant section. Suppose \( h \in G \). Then by property (v) of \( \sigma \),
\[ \sigma(\langle \theta, F(s, g) \rangle \cdot h)(s) = \langle \theta, a(s) \rangle \cdot h = \langle \theta, a(sh) \rangle. \]
But the first term of this equation
\[ = \sigma(\langle \theta, \gamma(s, h)^{-1} g^{-1} b(s, g^{-1}) \rangle)(s) \]
\[ = \sigma(\langle \theta, \gamma(s, h)^{-1} \gamma(s, g^{-1}) b(s, g^{-1}) \rangle)(s) \]
and by the corollary, since \( \gamma(s, h) \) is the adjoint of an isometric isomorphism, this \( = \langle \theta, \gamma(s, h)^{-1} a(s) \rangle \). Since \( E \) is separable, it follows [2, Theorem 8.17.2(c)] that \( \gamma(s, h)^{-1} a(s) = a(sh) \) for almost all \( s \), i.e. \( a(s) \) is an invariant section.

Thus, to complete the proof it suffices to show \( a(s) \in A_j \) for almost all \( s \). Let \( \{\theta_i\} \) be a countable dense subset of \( E \), considered as linear functionals on \( E^* \). Then the hyperplanes in \( E^* \) defined by \( \theta_i = q, q \) rational, separate all compact convex subsets of \( E^*_1 \) from points in \( E^*_1 \). Therefore, it suffices to show that for all \( \theta \) and \( q, \theta(A_j) > q \) implies \( \theta(a(s)) > q \) for almost all \( s \).
Given \( \theta \) and \( q \), let \( S_0 = \{s \in S | \theta(A_j) > q \} \). Then \( S_0 \) is measurable by [7, Lemma 1.7]. Suppose \( \mu(S_0) > 0 \). Then by property (iii) of \( \sigma \),
\[ \sigma(\langle \theta, F(s, g) \rangle \chi_{S_0 \times G}) > \sigma(q \cdot \chi_{S_0 \times G}) = q \chi_{S_0} \]
Thus \( \langle \theta, a(s) \rangle \cdot \chi_{S_0} > q \chi_{S_0} \) so \( \theta(a(s)) > q \) for almost all \( s \in S_0 \). Since \( \theta \) and \( q \) are arbitrary, the theorem follows.
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REFERENCES


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