CONVERGENCE OF CERTAIN COSINE SUMS IN THE METRIC SPACE $L_1$

BABU RAM

ABSTRACT. We consider here the $L_1$ convergence of Rees-Stanojević cosine sums to a cosine trigonometric series belonging to the class $S$ defined by Sidon and deduce as corollaries some previously known results from our result.

1. Introduction. Sidon [6] introduced the following class of cosine trigonometric series: Let

\begin{equation}
\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx
\end{equation}

be a cosine series satisfying $a_k = o(1)$, $k \to \infty$. If there exists a sequence $\{A_k\}$ such that

\begin{align}
&\text{(1.2)} \quad A_k \downarrow 0, \quad k \to \infty, \\
&\text{(1.3)} \quad \sum_{k=0}^{\infty} A_k < \infty, \\
&\text{(1.4)} \quad |\Delta a_k| < A_k, \quad \forall k,
\end{align}

we say that (1.1) belongs to the class $S$.

Let the partial sums of (1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n \to \infty} S_n(x)$.


\begin{equation}
g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx
\end{equation}

converge in the $L^1$ metric to (1.1) if and only if given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that

\begin{equation}
\int_{0}^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \epsilon
\end{equation}

for all $n > 0$. It has been shown in the same paper that the classical Young-Kolmogorov-Stanojević sufficient conditions for integrability of (1.1) imply (1.6).
CONVERGENCE OF COSINE SUMS

Generalising a classical result [1, p. 204], Teljakovskii [7] proved the following.

**Theorem A.** If (1.1) belongs to the class $S$, then a necessary and sufficient condition for $L^1$ convergence of (1.1) is $a_n \log n = o(1)$, $n \to \infty$.

2. **Lemmas.** The proofs of our results are based upon the following lemmas.

**Lemma 1** (Fomin [2]). If $|c_k| < 1$, then

$$
\int_0^\pi \left| \sum_{k=0}^n c_k \frac{\sin(k + 1/2)x}{2 \sin x/2} \right| dx < C(n + 1),
$$

where $C$ is a positive absolute constant.

**Lemma 2.** If (1.1) belongs to the class $S$, then

$$g_n(x) = S_n(x) - a_{n+1}D_n(x),$$

where $D_n(x)$ denotes the Dirichlet kernel.

**Proof.** Since (1.1) belongs to the class $S$, we have

$$a_k \to 0 \quad \text{and} \quad \sum_{k=0}^\infty |\Delta a_k| < \infty.$$

The conditions of Lemma 1 of Garrett and Stanojević [3] are thus satisfied and the result follows.

3. **Main result.** The main result of this paper reads:

**Theorem.** If (1.1) belongs to the class $S$, then (1.6) holds. Hence

$$\|f - g_n\|_{L^1} = o(1), \quad n \to \infty.$$

**Proof.** Making use of Abel’s transformation and Lemma 1, we have

$$
\int_0^\pi |f(x) - g_n(x)| \, dx = \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta a_k D_k(x) \right| \, dx
$$

$$= \int_0^\pi \left| \sum_{k=n+1}^\infty A_k \frac{\Delta a_k}{A_k} D_k(x) \right| \, dx
$$

$$= \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta A_k \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x) \right| \, dx
$$

$$\leq \sum_{k=n+1}^\infty \Delta A_k \int_0^\pi \left| \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x) \right| \, dx
$$

$$\leq C \sum_{k=n+1}^\infty (k + 1)\Delta A_k.$$

(1.2) and (1.3) now imply the conclusion of the Theorem.
4. **Corollaries.** (i) Using Lemma 2, we notice that

\[
\int_0^\pi |f(x) - S_n(x)| \, dx = \int_0^\pi |f(x) - g_n(x) + g_n(x) - S_n(x)| \, dx \\
\leq \int_0^\pi |f(x) - g_n(x)| \, dx + \int_0^\pi |g_n(x) - S_n(x)| \, dx \\
\leq \int_0^\pi |f(x) - g_n(x)| \, dx + \int_0^\pi |a_{n+1}D_n(x)| \, dx
\]

and

\[
\int_0^\pi |a_{n+1}D_n(x)| \, dx = \int_0^\pi |g_n(x) - S_n(x)| \, dx \\
\leq \int_0^\pi |f(x) - S_n(x)| \, dx + \int_0^\pi |f(x) - g_n(x)| \, dx.
\]

Since \(\lim_{n \to \infty} \int_0^\pi |f(x) - g_n(x)| \, dx = 0\) by our Theorem and \(\int_0^\pi |a_{n+1}D_n(x)| \, dx\) behaves like \(a_{n+1} \log n\) for large values of \(n\), Theorem A of Teljakovskiï follows.

(ii) Let \(a_k \to 0\) and \(\sum_{k=1}^{\infty} (k + 1)|\Delta^2 a_k| < \infty\). Then \(g_n\) converges to \(f\) in the metric space \(L\) since the trigonometric cosine series (1.1) with quasi-convex coefficients belongs to the class \(S\) if we choose \(A_k = \sum_{m=k}^{\infty} |\Delta^2 a_m|\). This is Example 1 of [3].

5. **Remark.** In [4], Garrett and Stanojević proved (Corollary B, p. 70) that their Theorem B extends the Teljakovskiï result.

My thanks are due to the referee for his wise comments which have definitely improved the presentation of this paper.

**References**


Department of Mathematics, Rohtak University, Rohtak-124001, India