C*-ALGEBRAS ISOMORPHIC AFTER TENSORING

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ABSTRACT. It is always true that whenever A and B are isomorphic C*-algebras then \( M_2 \otimes A \) and \( M_2 \otimes B \) are also isomorphic, and the converse holds for many standard examples. In this note we present two C*-algebras A and B such that \( M_2 \otimes A \) and \( M_2 \otimes B \) are isomorphic whereas A and B are not.

We remark that whenever A and B are isomorphic C*-algebras then \( N_n \otimes A \) and \( N_n \otimes B \) are also isomorphic, \( n = 1, 2, 3, \ldots \); the converse is true for abelian C*-algebras A and B because the center of \( N_n \otimes A \) (respectively \( N_n \otimes B \)) is isomorphic to A (respectively B). It follows also from the classification theory of [3] that if A and B are uniformly hyperfinite algebras such that \( M_2 \otimes A \) and \( M_2 \otimes B \) are isomorphic, then so are A and B. Finally, if A and B are perturbed block diagonal algebras then one verifies easily by applying [4, Theorem 1] that \( M_2 \otimes A \) isomorphic to \( M_2 \otimes B \) implies that A is isomorphic to B. The purpose of this note is to exhibit two C*-algebras A and B such that \( M_2 \otimes A \) is isomorphic to \( M_2 \otimes B \) but A is not isomorphic to B.

In what follows, \( \mathcal{L}(\mathcal{K}) \) (respectively \( \mathcal{C}(\mathcal{K}) \)) shall denote the algebra of bounded (respectively compact) operators on a separable Hilbert space \( \mathcal{K} \). By the essential commutant of a set \( S \subseteq \mathcal{L}(\mathcal{K}) \), denoted \( \mathcal{E}(S) \), we shall mean the set of \( T \in \mathcal{L}(\mathcal{K}) \) such that \( TS - ST \in \mathcal{C}(\mathcal{K}) \) for every \( S \in S \). A 2 \( \times \) 2 system of matrix units for a C*-algebra \( \mathcal{S} \) with identity I is defined to be a set \( \{ e_{ij} \}_{1 \leq i, j \leq 2} \) of elements of \( \mathcal{S} \) such that \( e_{ij} e_{km} = \delta_{jk} e_{im} \), \( e_{ij}^* = e_{ij} \), and \( e_{11} + e_{22} = I \). Finally, the natural projection of \( \mathcal{L}(\mathcal{K}) \) onto the Calkin algebra \( \mathcal{L}(\mathcal{K})/\mathcal{C}(\mathcal{K}) \) will be written \( \pi \). Let

\[
\begin{align*}
\mathcal{A} &= \{ T \oplus T : T \in \mathcal{L}(\mathcal{K}) \} + \mathcal{C}(\mathcal{K} \oplus \mathcal{K}), \\
\mathcal{B} &= \{ 0 \oplus T \oplus T : T \in \mathcal{L}(\mathcal{K}) \} + \mathcal{C}(\mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K}),
\end{align*}
\]

where \( \mathcal{K} \), the first coordinate space, is one dimensional.

To show that A and B are not isomorphic, we rely on the fact that two C*-algebras of \( \mathcal{L}(\mathcal{K}) \) which contain \( \mathcal{C}(\mathcal{K}) \) are isomorphic if and only if they

Received by the editors October 29, 1976.

AMS (MOS) subject classifications (1970). Primary 46L05, 47C10; Secondary 47B05, 47A55, 47A65.

Key words and phrases. C*-algebra, isomorphism, compact operators, essential commutant, matrix units, Hilbert space.

This work was partially supported by NSF contract #MCS 75-06482 A01.

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are unitarily equivalent, denoted by $\simeq$ [1, Theorem 1.3.4; Corollary 3, p. 20]. We note that because a unitary equivalence of two $C^*$-algebras induces a unitary equivalence of their essential commutants, to prove $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic, it suffices to prove that $E.C. (\mathcal{A})$ and $E.C. (\mathcal{B})$ are not.

It is easy to verify that:

$$E.C. (\mathcal{A}) = \left\{ \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} : \lambda_j \in \mathbb{C} \right\} + \mathcal{C}(\mathcal{K} \oplus \mathcal{K}),$$

$$E.C. (\mathcal{B}) = \left\{ 0 \oplus \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} : \lambda_j \in \mathbb{C} \right\} + \mathcal{C}(\mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K}),$$

where $\mathcal{K}$ and $\mathcal{K}$ are as before and $\mathbb{C}$ denotes the set of complex numbers.

In [2] it is shown that any $2 \times 2$ system of matrix units in $\pi(E.C.(\mathcal{A}))$ is the image of a $2 \times 2$ system of matrix units in $E.C. (\mathcal{A})$ whereas no $2 \times 2$ system of matrix units in $\pi(E.C.(\mathcal{B}))$ is, thus yielding the desired conclusion.

It remains only to show that $\mathcal{M}_2 \otimes \mathbb{A}$ and $\mathcal{M}_2 \otimes \mathbb{B}$ are isomorphic. In fact, we remark that slight modifications of our argument will show that $\mathcal{M}_n \otimes \mathbb{A}$ and $\mathcal{M}_n \otimes \mathbb{B}$ are isomorphic if and only if $n$ is even—although we do not need this here. An observation which we isolate to summon repeatedly is that $\mathcal{M}_n \otimes \mathcal{L}(\mathcal{K})$ (respectively $\mathcal{M}_n \otimes \mathcal{C}(\mathcal{K})$) is isomorphic to $\mathcal{L}(\mathcal{K})$ (respectively $\mathcal{C}(\mathcal{K})$) whenever $\mathcal{K}$ is an infinite dimensional Hilbert space.

So, we concretely represent $\mathcal{M}_2 \otimes \mathbb{B}$ as

$$\begin{pmatrix} 0 & T_{11} & 0 & T_{12} \\ T_{11} & 0 & T_{12} & 0 \\ 0 & T_{21} & 0 & T_{22} \\ T_{21} & 0 & T_{22} & 0 \end{pmatrix} + \mathcal{C}(\mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K})$$

which is unitarily equivalent to

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \oplus \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : \mathcal{T}_y \in \mathcal{L}(\mathcal{K})$$

by permuting the coordinate spaces. By employing the observation above, we realize (2) as unitarily equivalent to

$$\{ (0 \oplus T) \oplus (0 \oplus T) : 0 \in \mathcal{L}(\mathcal{K}), T \in \mathcal{L}(\mathcal{K}') \}$$

$$+ \mathcal{C}(\mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K}')$$

where $\mathcal{K}'$ is identified with $\mathcal{K} \oplus \mathcal{K}$. 
Because \( \dim (\mathcal{H}) < \infty \) and our algebra contains all of the compact operators on the underlying space, (3) is equal to

\[
\{ S \oplus S : S \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}') \} + \mathcal{C}(\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H}'),
\]

which by the observation mentioned above is unitarily equivalent to

\[
\left\{ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : T_{ij} \in \mathcal{L}(\mathcal{H}) \right\}
\]

\[+ \mathcal{C}(\mathcal{H} \oplus \mathcal{H} \oplus (\mathcal{H} \oplus \mathcal{H}')).\]

By interchanging the second and third coordinate spaces, we recognize (5) as

\[
\left\{ \begin{pmatrix} T_{11} & 0 & T_{12} \\ 0 & T_{11} & 0 \\ T_{21} & 0 & T_{22} \\ 0 & T_{21} & 0 \end{pmatrix} : T_{ij} \in \mathcal{L}(\mathcal{H}) \right\} + \mathcal{C}(\mathcal{H} \oplus \mathcal{H} \oplus (\mathcal{H} \oplus \mathcal{H})).
\]

which is the concrete representation of \( \mathfrak{M}_2 \otimes \mathfrak{A} \).

\[\square\]

**References**


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