A MEASURE THEORETIC VARIANT
OF BLUMBERG'S THEOREM

JACK B. BROWN

Abstract. It is the purpose of this note to present a measure theoretic
variant of Blumberg's theorem about continuous restrictions of arbitrary
real functions.

Henry Blumberg [1] proved a theorem in 1922, which in its most basic form
is stated as follows:

Theorem A. For every \( f: I \rightarrow \mathbb{R} \), there exists \( D \subset I \), \( D \) dense in \( I \), such that
\( f|D \) is continuous.

\( I \) is the interval \([0, 1]\) and \( \mathbb{R} \) denotes the reals. Blumberg observed [2] that
the set \( D \) cannot be made to have cardinality \( c \) because of the function \( f: I \rightarrow \mathbb{R} \) of Sierpiński and Zygmund [6] which has no continuous restriction of
cardinality \( c \). However, the author established theorems [3] from which the
following results:

Theorem B. For every \( f: I \rightarrow \mathbb{R} \), there exists \( W \subset I \), \( W \) c-dense in \( I \), such
that \( f|W \) is pointwise discontinuous (PWD).

\( W \) is c-dense in \( I \) if and only if every subinterval of \( I \) contains \( c \)-many
points of \( W \). A function \( g \) with domain \( W \) is PWD if and only if there exists
\( D \subset W \), \( D \) dense in \( W \), such that \( g \) is continuous at each element of \( D \).

P. Erdős recently asked the author if something could be done to make the
set \( W \) in the conclusion of Theorem B large relative to Lebesgue measure \( \lambda \).

First, notice that it would be easy to alter the set \( W \) in such a way to make
it an M set (i.e. have \( \lambda^0(W) > 0 \)). Let \( D \subset W \) be the set on which \( f|W \) is
continuous, let \( C \) be a Cantor subset of \( I \) of positive measure, and let
\( W' = W \cup C \). Then \( \lambda^0(W') > 0 \), and \( f|W' \) is still continuous at each
element of \( D - (C \cap D) \), which is dense in \( W' \).

On the other hand, it would not be possible to make \( W \) have outer measure
1 or even be M-dense in \( I \) if \( f \) is a function such as the following: let
\( C_1, C_2, \ldots, \) be a sequence of disjoint Cantor subsets of \( I \) such that \( C_1 \cup C_2 \cup \ldots \)
has measure 1, and let \( f(x) = n \) if \( x \in C_n \). \( A \) is M-dense in \( B \) if
\( A \subset B \) and every open set which intersects \( B \) intersects \( A \) in an M set.

Next we might ask if \( W \) can be made to be M-dense in itself and drop the

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requirement that it be dense in \( I \). Dropping the requirement that \( W \) be dense in \( I \) will, in fact, make it possible to obtain differentiability on a dense subset of \( W \) if infinite derivatives are allowed (i.e. \( f \) is differentiable at \( x \) if \( f \) is continuous at \( x \), \( x \) is a limit point of \( D_{f} \), and there is \( t, -\infty < t < \infty \), such that if \( \{x_{n}\} \) is a sequence of elements of \( D_{f} \) converging to \( x \), then \( \{(f(x) - f(x_{n}))/\left(x - x_{n}\right)\} \) converges to \( t \)). In [5] Ceder proved the following:

**Theorem C.** Suppose \( X \subseteq I \) is uncountable. Then for every \( f: X \to \mathbb{R} \) there exists \( D \subseteq X \), \( D \) bilaterally dense in itself, such that \( f|D \) is differentiable.

In [4] the author proved the following variant of Ceder’s theorem:

**Theorem D.** Suppose \( X \subseteq I \) is not an \( L_{1} \) set. Then for every \( f: X \to \mathbb{R} \) there exist \( W \subseteq X \) and \( D \subseteq W \), \( W \) bilaterally c-dense in itself and \( D \) dense in \( W \) such that \( f|W \) is differentiable at each element of \( D \).

An \( L_{1} \) set is a countable union \( M_{1} \cup M_{2} \cup \ldots \) such that for each \( i \), every nowhere dense in \( M_{i} \) subset of \( M_{i} \) has cardinality less than \( c \). A set is bilaterally dense (c-dense) (\( M \)-dense) in itself if and only if every closed interval which intersects it intersects it in an infinite set (set of cardinality \( c \)) (\( M \) set). The converse of Theorem D was also shown to hold. An alteration of the proof of Theorem D will prove the following:

**Theorem E.** Suppose \( X \subseteq I \) is an \( M \) set. Then for every \( f: X \to \mathbb{R} \), there exist \( W \subseteq X \) and \( D \subseteq W \), \( W \) bilaterally \( M \)-dense in itself and \( D \) dense in \( W \), such that \( f|W \) is differentiable at each element of \( D \).

The proof of Theorem E can proceed almost identically with the proof of Theorem 1 of [4], with the notions “\( L_{1} \)” and “\( L_{2} \)” (which means “not \( L_{1} \)” of [4] replaced by “null” (which in this case means “of measure zero”) and “\( M \)” respectively. Lemma 1 of [4] states that if \( x \) is an element of a bilaterally \( L_{2} \)-dense in itself set \( A \), then there exists a bilaterally c-dense in itself nowhere dense in \( A \) subset \( N \) of \( A \) containing \( x \). Indeed, the fact that the Continuum Hypothesis implies the existence of a Lusin set, so that the conclusion of Lemma 1 cannot be obtained if it is just known that \( A \) is bilaterally c-dense in itself, is the cause of most of the difficulties encountered in proving Theorems B and D, and brought about the necessity of defining properties \( L_{1} \) and \( L_{2} \). The situation with respect to \( M \) sets is much simpler in the sense that Lemma 1 of [4] can be replaced by the following:

**Lemma 1’.** If \( x \) is an element of a bilaterally \( M \)-dense in itself set \( A \), then there exists a bilaterally \( M \)-dense in itself nowhere dense in \( A \) subset \( N \) of \( A \) containing \( x \).

**Proof.** There exists a \( G_{2} \) set \( B \) such that \( A \subseteq B \) and \( \lambda^{0}(A) = \lambda(B) \). Assume that \( B \subseteq \text{Cl}(A) \), so that \( B \) is \( M \)-dense in itself. It follows that if \( C \subseteq B \) and \( \lambda(C) > 0 \), then \( \lambda^{0}(A \cap C) = \lambda(C) \). For each positive integer \( n \), the set \( A_{n} = [x, x + 1/n] \cap A \) has positive outer measure. Consider a Cantor set \( C_{n} \).
of positive measure such that $C_n$ is a subset of and nowhere dense relative to $[x, x + 1/n] \cap B$. Let $C_n'$ consist of the points of density $1$ of $C_n$. Then $R_n = C_n' \cap A$ will be a bilaterally $M$-dense in itself subset of $[x, x + 1/n] \cap A$ which is nowhere dense in $[x, x + 1/n] \cap A$. The set $[x - 1/n, x] \cap A$ will contain a similar set $L_n$. Then

$$N = L_1 \cup L_2 \cup \ldots \cup (x) \cup R_1 \cup R_2 \cup \ldots$$

is the desired set.

Now, Lemmas 2–6 of [4] can be altered by replacing “$L_1$” and “$L_2$” by “null” and “$M$”, respectively, and the proofs will be the same. Then on stage (2) of the inductive procedure in the proof of Theorem 1 [4, p. 39], make $N_s$ be bilaterally $M$-dense in itself, and Theorem E will be proved.

REFERENCES


Department of Mathematics, Auburn University, Auburn, Alabama 36830