A MEASURE THEORETIC VARIANT OF BLUMBERG'S THEOREM

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Abstract. It is the purpose of this note to present a measure theoretic variant of Blumberg's theorem about continuous restrictions of arbitrary real functions.

Henry Blumberg [1] proved a theorem in 1922, which in its most basic form is stated as follows:

**Theorem A.** For every \( f: I \to \mathbb{R} \), there exists \( D \subset I \), \( D \) dense in \( I \), such that \( f|D \) is continuous.

\( I \) is the interval \([0, 1]\) and \( \mathbb{R} \) denotes the reals. Blumberg observed [2] that the set \( D \) cannot be made to have cardinality \( c \) because of the function \( f: I \to \mathbb{R} \) of Sierpiński and Zygmund [6] which has no continuous restriction of cardinality \( c \). However, the author established theorems [3] from which the following results:

**Theorem B.** For every \( f: I \to \mathbb{R} \), there exists \( W \subset I \), \( W \) c-dense in \( I \), such that \( f|W \) is pointwise discontinuous (PWD).

\( W \) is c-dense in \( I \) if and only if every subinterval of \( I \) contains \( c \)-many points of \( W \). A function \( g \) with domain \( W \) is PWD if and only if there exists \( D \subset W \), \( D \) dense in \( W \), such that \( g \) is continuous at each element of \( D \).

P. Erdős recently asked the author if something could be done to make the set \( W \) in the conclusion of Theorem B large relative to Lebesgue measure \( \lambda \).

First, notice that it would be easy to alter the set \( W \) in such a way to make it an \( M \) set (i.e. have \( \lambda^0(W) > 0 \)). Let \( D \subset W \) be the set on which \( f|W \) is continuous, let \( C \) be a Cantor subset of \( I \) of positive measure, and let \( W' = W \cup C \). Then \( \lambda^0(W') > 0 \), and \( f|W' \) is still continuous at each element of \( D - (C \cap D) \), which is dense in \( W' \).

On the other hand, it would not be possible to make \( W \) have outer measure 1 or even be \( M \)-dense in \( I \) if \( f \) is a function such as the following: let \( C_1, C_2, \ldots \) be a sequence of disjoint Cantor subsets of \( I \) such that \( C_1 \cup C_2 \cup \ldots \) has measure 1, and let \( f(x) = n \) if \( x \in C_n \). \( A \) is \( M \)-dense in \( B \) if \( A \subset B \) and every open set which intersects \( B \) intersects \( A \) in an \( M \) set.

Next we might ask if \( W \) can be made to be \( M \)-dense in itself and drop the...
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requirement that it be dense in \( I \). Dropping the requirement that \( W \) be dense in \( I \) will, in fact, make it possible to obtain differentiability on a dense subset of \( W \) if infinite derivatives are allowed (i.e. \( f \) is differentiable at \( x \) if \( f \) is continuous at \( x \) and the limit \( -\infty < t < \infty \), such that if \( \{ x_n \} \) is a sequence of elements of \( D \) converging to \( x \), then \( \left( \frac{f(x) - f(x_n)}{x - x_n} \right) \) converges to \( t \). In [5] Ceder proved the following:

**Theorem C.** Suppose \( X \subset I \) is uncountable. Then for every \( f: X \to R \) there exists \( D \subset X \) such that \( f|D \) is differentiable.

In [4] the author proved the following variant of Ceder’s theorem:

**Theorem D.** Suppose \( X \subset I \) is not an \( L_1 \) set. Then for every \( f: X \to R \) there exist \( W \subset X \) and \( D \subset W \) such that \( f|W \) is differentiable at each element of \( D \).

An \( L_1 \) set is a countable union \( M_1 \cup M_2 \cup \ldots \) such that for each \( i \), every nowhere dense in \( M_i \) subset of \( M_i \) has cardinality less than \( c \). A set is bilaterally dense (\( c \)-dense) (\( M \)-dense) if and only if every closed interval which intersects it intersects it in an infinite set (set of cardinality \( c \)) (\( M \) set). The converse of Theorem D was also shown to hold. An alteration of the proof of Theorem D will prove the following:

**Theorem E.** Suppose \( X \subset I \) is an \( M \) set. Then for every \( f: X \to R \), there exist \( W \subset X \) and \( D \subset W \) such that \( f|W \) is differentiable at each element of \( D \).

The proof of Theorem E can proceed almost identically with the proof of Theorem 1 of [4], with the notions “\( L_1 \)” and “\( L_2 \)” (which means “not \( L_1 \)” of [4] replaced by “null” (which in this case means “of measure zero”) and “\( M \)”, respectively. Lemma 1 of [4] states that if \( x \) is an element of a bilaterally \( L_2 \)-dense in itself set \( A \), then there exists a bilaterally \( c \)-dense in itself nowhere dense in \( A \) subset \( N \) of \( A \) containing \( x \). Indeed, the fact that the Continuum Hypothesis implies the existence of a Lusin set, so that the conclusion of Lemma 1 cannot be obtained if it is just known that \( A \) is bilaterally \( c \)-dense in itself, is the cause of most of the difficulties encountered in proving Theorems B and D, and brought about the necessity of defining properties \( L_1 \) and \( L_2 \). The situation with respect to \( M \) sets is much simpler in the sense that Lemma 1 of [4] can be replaced by the following:

**Lemma 1’.** If \( x \) is an element of a bilaterally \( M \)-dense in itself set \( A \), then there exists a bilaterally \( M \)-dense in itself nowhere dense in \( A \) subset \( N \) of \( A \) containing \( x \).

**Proof.** There exists a \( G_\delta \) set \( B \) such that \( A \subset B \) and \( \lambda^0(A) = \lambda(B) \). Assume that \( B \subset \text{Cl}(A) \), so that \( B \) is \( M \)-dense in itself. It follows that if \( C \subset B \) and \( \lambda(C) = 0 \), then \( \lambda^0(A \cap C) = \lambda(C) \). For each positive integer \( n \), the set \( A_n = \{x, x + 1/n \} \cap A \) has positive outer measure. Consider a Cantor set \( C_n \).
of positive measure such that \( C_n \) is a subset of and nowhere dense relative to \([x, x + 1/n] \cap B\). Let \( C_n' \) consist of the points of density 1 of \( C_n \). Then \( R_n = C_n' \cap A \) will be a bilaterally \( M \)-dense in itself subset of \([x, x + 1/n] \cap A\) which is nowhere dense in \([x, x + 1/n] \cap A\). The set \([x - 1/n, x] \cap A\) will contain a similar set \( L_n \). Then

\[
N = L_1 \cup L_2 \cup \ldots \cup \langle x \rangle \cup R_1 \cup R_2 \cup \ldots
\]

is the desired set.

Now, Lemmas 2–6 of [4] can be altered by replacing "\( L_1 \)" and "\( L_2 \)" by "null" and "\( M \)", respectively, and the proofs will be the same. Then on stage (2) of the inductive procedure in the proof of Theorem 1 [4, p. 39], make \( N \) be bilaterally \( M \)-dense in itself, and Theorem E will be proved.

REFERENCES