A NOTE ON $k$-CRITICALLY $n$-CONNECTED GRAPHS

R. C. ENTRINGER AND PETER J. SLATER

ABSTRACT. A graph $G$ is said to be $(n^*, k)$-connected if it has connectivity $n$ and every set of $k$ vertices is contained in an $n$-cutset. It is shown that an $(n^*, k)$-connected graph $G$ contains an $n$-cutset $C$ such that $G - C$ has a component with at most $n/(k + 1)$ vertices, thereby generalizing a result of Chartrand, Kaugars and Lick. It is conjectured, however, that $n/(k + 1)$ can be replaced with $n/2k$ and this is shown to be best possible.

The terminology and notation of [1] will be used throughout. In [4] the notion of a critically connected graph was generalized as follows. A graph with connectivity $n$ is $k$-critical if whenever $S$ is a vertex set with $|S| < k$ the connectivity of $G - S$ is $n - |S|$. That is, every vertex set with no more than $k$ members is contained in an $n$-cutset or a trivializing set of $n$ vertices. Such a graph will be said to be $(n^*, k)$-connected.

Chartrand, Kaugars, and Lick have shown [2] that an $(n^*, 1)$-connected graph, $n > 2$, contains a vertex of degree at most $3n/2 - 1$. This would follow, of course, if it were known that an $(n^*, 1)$-connected graph $G$ contained an $n$-cutset $C$ such that $G - C$ contained a component with at most $n/2$ vertices. Our object in this note is the generalization of the latter statement. To this end we develop the following notation.

Given a graph $G$ with connectivity $n$ let $C_G$ be the family of all $n$-cutsets of $G$. If $C$ is a member of $C_G$ denote by $v(C)$ the number of vertices in a smallest component of $G - C$. Finally, let $r(G) = \min v(C)$ where the minimum is taken over all members $C$ of $C_G$. That is, $r(G)$ is the order of a smallest component that can be obtained by removal of an $n$-cutset from $G$.

THEOREM. If $G$ is an $(n^*, k)$-connected graph with $1 < k < n$ then $r(G) < n/(k + 1)$.

PROOF. Choose an $n$-cutset $C$ of $G$ so that $v(C) = r = r(G)$, and let $R$ be the vertex set of a component of $G - C$ chosen so that $|R| = r$. Our proof will follow the following property of $R$:

(i) An $n$-cutset $D$ of $G$ that contains a vertex of $R$ contains all vertices of $R$.

To show this we let $L$ be the vertex set of $G - C - R$, let $T$ be the vertex set...
of one component of $G - D$ and let $B$ be the vertex set of the remaining components of $G - D$ (see Figure 1).

Since it is easy to argue that if $L \cap T$ is not empty then $(L \cap D) \cup (C \cap D) \cup (C \cap T)$ is a cutset of $G$, we suppress the details and note that analogous results hold for $L \cap B$, $R \cap T$, and $R \cap B$.

We can now conclude that at least one of the sets $L \cap T$ and $R \cap B$ is empty. If this were not so we must have $|L \cap D) \cup (C \cap D) \cup (C \cap T)| > n$ and $|(R \cap D) \cup (C \cap D) \cup (C \cap B)| > n$. But equality must hold for both expressions since $|C| = |D| = n$. This is impossible, however, since it implies $r < |R \cap B| < |R|$. By similar argument one of the sets $L \cap B$ and $R \cap T$ must be empty.

Now, if (i) is not true, we may assume $R \cap T \neq \emptyset$ so that $L \cap B = \emptyset$. If, also, $R \cap B \neq \emptyset$ then $L \cap T = \emptyset$ so that $|R \cap D| < |R| < |L| = |L \cap D|$. Consequently, $|R \cap D| < \frac{1}{2}(n - |C \cap D|)$, which, in turn, gives

$$2n < |C \cap T| + 2|C \cap D| + 2|R \cap D| + |C \cap B| < |C| + n.$$  

Since this is impossible we must have $R \cap B = \emptyset$ so that $|C \cap B| = |B| > |R| > |R \cap D|$. But this implies $(C \cap T) \cup (C \cap D) \cup (R \cap D)$ is a cutset with fewer than $n$ vertices. Consequently (i) is proven and we now show:

(ii) The theorem holds for $k = 1$.

We assume otherwise and, referring to Figure 1, note that $|R \cap D| = |R| > n/2$ implies $|L \cap D| < |(L \cap D) \cup (C \cap D)| < n/2$, so that we assume $L \cap T \neq \emptyset$ and, consequently, must have $|C \cap T| > n/2$. This implies $|C \cap B| < n/2$ so that $L \cap B$ cannot be empty. But then $(L \cap D) \cup (C \cap D) \cup (C \cap B)$ is a cutset with fewer than $n$ vertices, which cannot be. Hence (ii) holds and we now show:

(iii) $G - R$ is an $((n - r)^*, k - 1)$-connected graph.
$G - R$ is obviously $(n - r)$-connected. To show that any $k - 1$ vertices of
$G - R$ lie in an $(n - r)$-cutset of $G - R$ we choose any such $(k - 1)$ set $S$ of
$G - R$ together with one vertex $p$ of $R$ and extend this to an $n$-cutset $S'$ of $G$.
By (i) $R$ is a subset of $S'$ so that $S' - R$ is an $(n - r)$-cutset of $G - R$
containing $S$, and (iii) is proven.

We can now complete the proof of the theorem by induction on $k$. We may
assume it holds for all $(k - 1)$-critically connected graphs with $k > 1$. Then,
by (iii), $G - R$ contains an $(n - r)$-cutset $C'$ such that $G - R - C'$ has a
component $R'$ with at most $(n - r)/k$ vertices. But since $R \cup C'$ is an
$n$-cutset of $G$ we must have $r < (n - r)/k$, i.e., $r < n/(k + 1)$, and the proof
is complete.

We obtain the following consequence immediately upon consideration of
the degree of a vertex in the set $R$ described in the proof of the theorem.

Corollary. An $(n^*, k)$-connected graph contains a vertex of degree at most
$(k + 2)n/(k + 1) - 1$.

In particular, we can conclude that any $(n^*, k)$-connected graph with
$k > (n - 1)/2$ has a vertex of degree $n$. In the cases $n = 2$ and $3$ more is
known, however. L. Nebesky [5] has shown that a $(2, 1)$-connected graph with
at least six vertices contains four vertices of degree $2$ and this result is best.
The authors [3] have shown that a $(3, 1)$-connected graph contains at least
two vertices of degree $3$ and this result is best. Suppose, now, that $G$ is a
$(4, 2)$-connected graph. Then $G$ has a vertex $p$ of degree $4$ so that, by (iii)
above, $G - p$ is a $(3, 1)$-connected graph and consequently has two vertices
of degree $3$. $G$, then, had at least three vertices of degree $4$. The possibility
that such properties are not restricted to graphs with low connectivity can be
made explicit as follows.

Conjecture 1. An $(n, k)$-connected graph with $k > (n - 1)/2$ contains at
least two vertices of degree $n$.

We do not believe that the result of the theorem is best except at $k = 1$ and
$k = [n/2]$, but, however, do have some confidence in the following conjecture.

Conjecture 2. If $G$ is an $(n^*, k)$-connected graph then $r(G) < n/2k$.

We will describe a class of graphs, Figure 2, showing that this conjecture, if
true, is best possible. For each $k > 1$ and $r > 1$ we define a graph $G_{k,r}$ as
follows. The vertex set of $G_{k,r}$ consists of $2k + 2$ sets $S_1, \ldots, S_{2k+2}$ of $r$
vertices each. Two vertices are adjacent if and only if they lie in sets $S_i$ and $S_j$
such that $i - j \equiv k + 1 \mod(2k + 2)$. It is obvious that $G_{k,r}$ has connectivity
$2kr$ and that $r(G_{k,r}) = r$. Also, for any choice of a set $S$ of $k$ vertices of $G_{k,r}$
there will be an $i$ such that neither $S_i$ or $S_{i+k+1}$ (indices reduced $\mod(2k + 1)$)
contains a vertex of $S$. Consequently, $S$ can be completed to a $2kr$-cutset
and so $G_{k,r}$ is a $((2kr)^*, k)$-connected graph.

Conjecture 2, in addition to being correct, with proper interpretation, for
noncritical graphs, i.e. at $k = 0$, would imply that if $G$ is an $(n, k)$-connected
graph then either $k \leq \lfloor n/2 \rfloor$ or $k = n$ and $G = k_{n+1}$. This latter implication has been conjectured by Slater [4].

Note added in proof. W. Mader has kindly informed us that property (i) in the proof of the Theorem had been previously proven by him [Eine eigenschaft der atome endlicher graphen, Arch. Math. (Basel) 22 (1971), 333–336.]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87131

SANDIA LABORATORIES, ALBUQUERQUE, NEW MEXICO 87115