ON THE NORMABILITY OF THE INTERSECTION
OF $L_p$ SPACES

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Abstract. The set $L_\omega = \bigcap_{p=1}^{\infty} L_p[0, 1]$ is not equal to $L_\infty[0, 1]$ since $L_\omega$ contains the function $-\ln x$. Using the theory of $L_p$ spaces for finitely additive set functions developed by Leader [9] we will prove several necessary and sufficient conditions for the normability of a generalization of $L_\omega$. These include the equality and finite dimensionality of all the $L_p$ spaces, $p > 1$.

1. Introduction. The set $L_\omega = \bigcap_{p=1}^{\infty} L_p[0, 1]$ is not equal to $L_\infty[0, 1]$ since $L_\omega$ contains the function $-\ln x$. In [2] Arens showed that with the natural $\omega$-topology, i.e. the smallest topology which contains each relative $p$-norm topology, $L_\omega$ is not normable. In [5] and [6] Davis, Murray and Weber discussed $L_p(\mu) = \cap_{q<p} L_q(\mu)$ which, when $p = \infty$ and $\mu$ is Lebesgue measure on $[0, 1]$, reduces to $L_\omega$. Using the theory of $L_p$ spaces for finitely additive set functions developed by Leader [9], we will prove several necessary and sufficient conditions for the normability of a generalization of $L_\omega$. These include the equality and finite dimensionality of all the $L_p$ spaces, $p > 1$.

Suppose $S$ is a set, $F$ a field of subsets of $S$ and $ba(F)$ the set of bounded finitely additive functions from $F$ into the real numbers. If $G \in ba(F)$, then $G^+$ will denote the nonnegative valued elements of $G$. $A_\lambda$ will be the set of elements of $ba(F)$ which are absolutely continuous with respect to $\lambda \in ba(F)^+$. It should be noted that we mean absolutely continuous in the $\varepsilon-\delta$ sense which is (for $ba(F)$) stronger than the $0-0$ sense. Henceforth, $\mu$ will be in $ba(F)^+$ with $\mu(S) = 1$. We adopt the convention that $a/0$ is $0$.

2. "$L_p$ spaces" for $\mu$ and $H_\omega(\mu)$. For $1 \leq p < \infty$ we define $H_p(\mu)$ to be the set to which $\xi$ belongs iff

(1) $\xi \in A_\mu$ and
(2) there exists an $M > 0$ such that $\sum_D |\xi(I)|^p \mu(I)^{1/p} < M$ for each (finite) subdivision $D$ of $S$ by elements of $F$.

If $p > 1$, then (1) may be replaced by: $\mu(I) = 0$ implies $|\xi(I)| = 0$. If $p = 2$
we have the Hilbert space of Hellinger integrable functions which, when \( F \) is a \( \sigma \)-field and \( \mu \) is countably additive, corresponds to \( L_2(\mu) \). The \( p \)-norm (\( 1 < p < \infty \)) denoted, \( \| \cdot \|_p \), is the \( p \)th root of the supremum of the sums in (2). \( H_\infty(\mu) \) will be \( \{ \xi \in A_\mu | \| \xi(I) \|_\mu(I)^{-1} \text{ is bounded on } F \} \) (the Lip(\( \mu \)) of [4]) with \( \| \xi \|_\infty = \sup \{ |\xi(I)| \| \mu(I)^{-1} | I \in F \} \). With this norm \( H_\infty(\mu) \) is a Banach space [9].

For \( \xi, \eta \) and \( \delta \) in \( ba(\mathcal{F}) \) we will denote by \( \eta \land \delta \) the element of \( ba(\mathcal{F}) \) whose value at \( v \in \mathcal{F} \) is \( \sup \{ g(v) | g \in ba(\mathcal{F}), g \leq \eta \text{ and } g \leq \delta \} \), \( \eta \lor \delta \) will be \( -[(-\eta) \land (-\delta)] \) and \( \xi \lor 0 \) will be written \( \xi^+ \), so that the total variation function for \( \xi \) will be \( |\xi| = |\xi^+ + \xi^-| \), where \( \xi^- = (-\xi)^+ \).

In the interest of self-containment we state some of the results from [9] which we will need.

**Theorem 2.1.** Suppose each of \( \eta \) and \( \delta \) is in \( A_\mu \), \( \lambda \in A_\mu^+ \) and \( 1 < q < p < \infty \). Then we have;

I. If \( |\delta| \leq |\eta| \), then \( \eta \in H_p(\mu) \iff |\eta| \in H_p(\mu) \), in which case \( \delta \in H_p(\mu) \) and \( \|\delta\|_p \leq \|\eta\|_p = \|\lambda\|_p \).

II. The sequence \( (\lambda \land K\mu)_{K=1}^\infty \) is \( p \)-norm convergent to \( \lambda \) iff \( \lambda \in H_p(\mu) \) and, therefore, \( H_\infty(\mu) \) is \( p \)-norm dense in \( H_p(\mu) \).

III. The strength of the \( p \)-norms is nondecreasing with \( p \) and therefore

\[
H_\infty(\mu) \subseteq H_p(\mu) \subseteq H_q(\mu) \subseteq A_\mu.
\]

Suppose \( \lambda \in A_\mu^+ \), \( 1 < p < \infty \), \( D \) is a subdivision of \( S \) and \( N \) is the number of elements in \( D \). Then there exists a \( K \in \mathbb{N} = \{1, 2, \ldots \} \) such that \( \lambda(v)^p < (\lambda \land K\mu)(v)^p + \mu(v)^p - 1 N^{-1} \) for each \( v \in D \). Therefore

\[
\sum_D \lambda(v)^p \mu(v)^{1-p} < \sum_D \left[ (\lambda \land K\mu)(v)^p + \mu(v)^p - 1 N^{-1} \right]^\mu(v)^{1-p}
\]

and we have

**Lemma 2.1.** If \( \lambda \in A_\mu^+ \) and \( 1 < p < \infty \), then \( \lambda \in H_p(\mu) \iff |\lambda| \leq K\mu \) for each \( K \in \mathbb{N} = \{1, 2, \ldots \} \) is bounded.

We will denote by \( H_\omega(\mu) \) the intersection of \( H_p(\mu) \) for \( 1 < p < \infty \). The \( \omega \)-topology will be the smallest topology on \( H_\omega(\mu) \) which contains each relative \( p \)-norm topology.

**Lemma 2.2.** If \( (a_p)_{p=1}^\infty \) is a sequence of positive numbers such that \( \sum_{p=1}^\infty a_p \|\xi\|_p \) exists for each \( \xi \) in \( H_\omega(\mu) \), then the norm defined by \( \|\xi\| = \sum_{p=1}^\infty a_p \|\xi\|_p \) is complete.

**Proof.** Let \( (\xi_n) \) be a \( \| \cdot \| \) Cauchy sequence in \( H_\omega(\mu) \). Then by 2.L.II we have

\[
\| \xi_n - \xi_m \| = \| \| \xi_n - \xi_m \| \leq \| \xi_n - \xi_m \| = \| \xi_n - \xi_m \|
\]

so that \( (\xi_n) \) is also \( \| \cdot \| \) Cauchy. Let \( (\delta_n) = (\|\xi_n\|) \) be a subsequence of \( (\|\xi_n\|) \)
such that if $i$ and $r$ are positive integers then $\|\delta_i - \delta_{i+r}\| < 1/2^r$. Let $\eta_1 = \delta_1$ and for each $1 < i \in \mathbb{N}$ let $\eta_i = \delta_i \vee \eta_{i-1}$. Then

$$0 < \eta_{i+1} - \eta_i = (\delta_{i+1} - \eta_i)^+ < (\delta_{i+1} - \delta_i)^+ < \|\delta_{i+1} - \delta_i\| < 1/2^i.$$ 

so that again by 2.L.II, $\|\eta_{i+1} - \eta_i\| < \|\delta_{i+1} - \delta_i\| < 1/2^i$. Now for each $i, r \in \mathbb{N}$ we have

$$\|\eta_{i+r} - \eta_i\| < \sum_{j=1}^{r} \|\eta_{i+j} - \eta_{i+j-1}\| < \sum_{j=1}^{r} \frac{1}{2^{j+1}} < \frac{1}{2^i}$$

and, therefore, $(\eta_i)$ is $\| \cdot \|$ Cauchy and $\eta_i < \eta_{i+1}$ for each $i \in \mathbb{N}$.

Now any norm Cauchy sequence is $p$-norm Cauchy for each $1 < p < \infty$ and, hence, has a $p$-norm limit which (by the comparability of $p$-norms) is independent of $p$. Therefore there exists $\xi$ and $\eta$ in $H_\infty(\mu)$ such that $\eta_i \to^p \eta$ and $\xi_i \to^p \xi$ for each $1 \leq p < \infty$.

Now let $c > 0$ and $K \in \mathbb{N}$ be such that $\Sigma_{k=1}^{\infty} a_p \|\eta\|_p < c/4$ and $\Sigma_{k=1}^{\infty} a_p \|\xi\|_p < c/4$. For each $p < K$ let $N_p$ be such that if $i > N_p$, then $a_p \|\xi - \xi_n\|_p < c/2K$. Let $N = \max\{N_1, N_2, \ldots, N_K\}$ and $i > N$. Then

$$\|\xi - \xi_n\| = \sum_{p=1}^{\infty} a_p \|\xi - \xi_n\|_p$$

$$= \sum_{p=1}^{K} a_p \|\xi - \xi_n\|_p + \sum_{p=K+1}^{\infty} a_p \|\xi - \xi_n\|_p$$

$$< \sum_{p=1}^{K} \frac{c}{2K} + \sum_{p=K+1}^{\infty} a_p (\|\xi\|_p + \|\xi_n\|_p)$$

$$< \frac{c}{2} + \frac{c}{4} + \sum_{p=K+1}^{\infty} a_p \|\delta\|_p$$

$$< \frac{3c}{4} + \sum_{p=K+1}^{\infty} a_p \|\eta\|_p$$

$$< \frac{3c}{4} + \sum_{p=K+1}^{\infty} a_p \|\eta\|_p < c.$$ 

Therefore, $\xi_n \to^p \|\xi$ and, hence, $\xi_n \to^p \|\xi$, so that $\| \| \|$ is complete.

3. A differential equivalence theorem for Hellinger integrals. In this section we will show that $H_{2k}(\mu)$ is the "$K$th image" of a certain function. The integral considered below is the refinement limit of sums over finite subdivisions of $S$ by elements of $F$. For further details see [1].

**Theorem 3.H.** If $\xi \in ba(F)$, then $\xi \in A_\mu^{++}$ iff there exists an $\eta \in H_2(\mu)$ such that $\xi = \int \eta^2/\mu$.

The proof of this theorem of Hellinger [8] (for interval functions) carries over to our setting and suggests the function

$$T: H_2^+(\mu) \to A_\mu^{++}: \eta \to \int \eta^2/\mu.$$
The $\eta$ of Theorem 3.H is given by the function $R(\xi) = f(\xi, \mu)^{1/2}$ which has the following properties [3]. $R$ is defined on all of $\text{ba}(F)^+$ and its restriction to $A^+_\mu$ is the inverse of $T$. If $\xi \in \text{ba}(F)^+$, then $R(\xi) \in H^+_2(\mu)$ and $T(R(\xi)) = a^+_\mu(\xi)$ (the absolutely continuous part of $\xi$).

LEMMA 3.1. Suppose $\alpha: F \to \mathbb{R}^+$, $\mu(I) = 0$ implies $\alpha(I) = 0$ for each $I \in F$, $\Sigma_D \alpha(I)$ is nondecreasing for successive refinements and $\int S\alpha(I)$ exists. Then $\int \alpha^2/\mu$ exists iff $\int (\alpha^2)^2/\mu$ exists, in which case they are equal.

PROOF. If $\int (\alpha^2)^2/\mu$ exists, then, since $\Sigma_D \alpha^2(I)/\mu(I)$ is nondecreasing for successive refinements and bounded by $\int_S (f\alpha(I))^2/\mu(v)$, we have $\int \alpha^2/\mu$ exists.

If $\int \alpha^2/\mu$ exists, then
\[
\int \alpha = \int \left( \frac{\mu \alpha^2}{\mu} \right)^{1/2} = \int \left( \frac{\mu \alpha^2}{\mu} \right)^{1/2} \quad \text{(see [1])}
\]
\[
= R \left( \int \alpha^2/\mu \right) \in H_2(\mu),
\]
i.e. $\int (\alpha^2)^2/\mu$ exists.

If both exist, then
\[
\int \alpha^2/\mu \leq \int \left( \int \alpha \right)^2/\mu = T \left( \int \alpha \right) = T \left( R \left( \int \alpha^2/\mu \right) \right) = a^+_\mu \left( \int \alpha^2/\mu \right) \leq \int \alpha^2/\mu.
\]

For $\xi \in \text{ba}(F)$ and $p > 1$ we have $\Sigma_D \xi^p(I)\mu(I)^{1-p}$ is nondecreasing for successive refinements [9] so that, for $\xi \in A^+_\mu$, $\xi \in H^p(\mu)$ iff $\int \xi^p\mu^{1-p}$ exists. Therefore by 3.1 and induction we have, for $\xi \in A^+_\mu$, $T^K(\xi)$ exists iff $\xi \in H_2^p(\mu)$. Hence $R^K(A^+_\mu) = H_2^p(\mu)$.

4. The normability of $H_\omega(\mu)$. In the countably additive case the equivalence of (2) and (3) below can be found in [5].

THEOREM 4.1. These are equivalent:
(1) The $\omega$-topology is contained in a norm topology.
(2) The $\omega$-topology is normable.
(3) There exists a $p$ ($1 < p < \infty$) such that $H_p(\mu) = H_\omega(\mu)$.
(4) $A^+_\mu = H_\omega(\mu)$.
(5) $A^+_\mu = H_\infty(\mu)$.
(6) $A^+_\mu$ is finite dimensional.

(1) $\to$ (2). Suppose $\|\cdot\|$ is a norm on $H_\omega(\mu)$ and its topology contains the $\omega$-topology. Then for each $p \in \mathbb{N}$ there exists an $M_p > 0$ such that $\|\xi\|_p \leq M_p \|\xi\|$ for each $\xi \in H_\omega(\mu)$ and, therefore, $\|\cdot\| = \sum_{p=1}^\infty \|\cdot\|_p 2^{-p}M_p^{-1}$ is a complete norm by Lemma 2.2. The $\omega$-topology, being defined by a countable collection of norms, is a complete linear metric topology which is contained in the $\|\cdot\|$ topology and, hence, equal to it by the open mapping theorem [7].

(2) $\to$ (3). Suppose the $\omega$-topology is normable, then there exists a neighborhood $U$ of 0 which is $\omega$-bounded in the following sense: if $(\eta_n) \subseteq U$ and
(αₙ) is a null sequence, then (αₙ ηₙ → 0). Since the ω-topology is generated by the p-norms, p ∈ N, there exists a p > 1 such that if B = {η ∈ Hₚ(μ) | \|η\|ₚ < 1}, then B ∩ H₀(μ) is ω-bounded. Now let ξ ∈ B⁺, r > p and ξₖ = ξ ∩ Kμ for each K ∈ N. Then \|ξₖ\|ᵣ is bounded, since otherwise we would have \((1/\|ξₖ\|ᵣ)₁/₂ → 0\) and, therefore, \((1/\|ξₖ\|ᵣ)₁/₂ ξₖ → 0\), hence \((1/\|ξₖ\|ᵣ)₁/₂ ξₖ \) r-norm convergent to 0, which is a contradiction. Therefore, by Lemma 2.1, ξ ∈ Hₚ(μ) and, hence, Hₚ(μ) = H₀(μ).

(3) → (4). Suppose p > 1 and H₀(μ) = Hₚ(μ). By 2.L. it is sufficient to consider A⁺. Let K ∈ N be such that p < 2^K, then by §3 we have

\[
A⁺ₚ = T^K (R^K (A⁺ₚ)) = T^K (H⁺₂ (μ)) = T^K (H⁺₂ (μ))
\]

We will need the following to prove that (4) implies (5).

**Theorem 4.2.** If each of λ and μ is in ba(F)+, then λ ∈ H₀(μ) iff H⁺₂(λ) ⊆ H₂(μ).

**Proof.** [4].

(4) → (5). Suppose H₂(μ) = A⁺ and λ ∈ A⁺. Then H₂(λ) ⊆ A⁺ ⊆ A⁺ = H₂(μ) and, therefore, by 4.B, λ ∈ H₀(μ).

(5) → (6). Suppose A⁺ is not finite dimensional; then there exists a disjoint sequence, (vₙ), on which μ is positive. Let (aₙ) be an unbounded sequence of positive numbers such that \(\sum_{n=1}^{∞} aₙ μ(vₙ) < ∞\) and \(\sum_{n=1}^{∞} aₙ μ_n < ∞\). Then \(\sum_{n=1}^{∞} aₙ μ_n< ∞\) is in A⁺ but not H₀(μ), where μₙ is the contraction of μ to vₙ, i.e. μₙ(I) = μ(I ∩ vₙ) for each I ∈ F.

(6) → (1). This follows from the fact that \(\|\cdot\|₂\) is stronger than each relative p-norm.

We conclude by noting that as a consequence of the equivalence of (1), (2) and (5) we have the following “internal” characterization of the normability of H₀(μ).

**Corollary 4.1.** The ω-topology is normable iff H₀(μ) = H₀(μ).

**Added in Proof.** For countably additive μ on a σ-field the property H₀(μ) = H₀(μ), 4.1.5, 4.1.6 and several other conditions have been considered by Professor Ion Chitescu in *Finitely purely atomic measures and Lᵖ-spaces*, Anal. Univ. București Sti. Natur. 24 (1975), 23–29, MR 52 #8366.

**Bibliography**


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