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ON THE SUPPORT OF SYMMETRIC INFINITELY DIVISIBLE AND STABLE PROBABILITY MEASURES ON LCTVS

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ABSTRACT. It is shown that the topological support (supp.) of a τ-regular, symmetric, infinitely divisible (resp. stable of any index $\alpha \in (0, 2)$) probability measure on a Hausdorff LCTVS E is a subgroup (resp. a subspace) of E. The part regarding the support of a stable probability measure of this theorem completes a result of A. De-Acosta [Ann. of Probability **3** (1975), 865–875], who proved a similar result for $\alpha \in (1, 2)$, and the author [Proc. Amer. Math. Soc. **63** (1977), 306–312], who proved it for $\alpha \in [1, 2)$. Further, it provides a complete affirmative solution to the question, raised by J. Kuelbs and V. Mandrekar [Studia Math. **50** (1974), 149–162], of whether the supp. of a symmetric stable probability measure of index $\alpha \in (0, 1]$ on a separable Hilbert space H is a subspace of H.

1. **Introduction.** The main purpose of this paper is to prove that the topological support (supp.) of a symmetric τ-regular infinitely divisible (i.d.) (respectively, stable of any index $\alpha \in (0, 2)$) probability measure (p.m.) on a locally convex topological vector space (LCTVS) E is a (closed) subgroup (respectively, a subspace) of E (Theorems 1, 2). The part regarding the support of a stable p.m. of this result completes a result of De-Acosta [4], who proved a similar result for $\alpha \in (1, 2)$, and the author [6], who proved it for $\alpha \in [1, 2)$. It also settles a question raised by Kuelbs and Mandrekar [5].

2. **Preliminaries.** All topological spaces considered are assumed Hausdorff and all measures are defined on the Borel σ-algebra (the σ-algebra generated by the class of open sets) of the topological space. All vector spaces considered are over the real field R. If A is a subset of a topological space E, then $\bar{A}$ will denote, throughout, the E-closure of A.

DEFINITION 1. Let μ be a finite measure on a topological space E. Then μ is called *τ-regular* if $\mu(\bigcup_\alpha O_\alpha) = \lim_\alpha \mu(O_\alpha)$ for every increasing net $\{O_\alpha\}$ of open sets of E. The unique smallest closed set with full μ-measure (if it exists) is called the *supp.* of μ. It is easy to show that whenever the supp. of a finite measure exists, then it is the set of those points of the space, every open neighborhood (nbd.) of which has positive μ-measure. Also, if μ is τ-regular

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then the supp. of μ always exists. If μ denotes a finite measure on a topological space, then, throughout, S_μ will denote the supp. of μ.

DEFINITION 2. Let μ and ν be two τ-regular finite measures on a LCTVS E; then, according to [1] and [2], the convolution of μ and ν, denoted by $\mu * \nu$, is defined by

$$\mu * \nu(A) = \int_E \mu(A - x)\nu(dx) = \int_E \nu(A - x)\,\mu(dx),$$

for every Borel set A.

DEFINITION 3. Let μ be a τ-regular p.m. on a LCTVS E; then μ is called i.d. if for each $n = 1, 2, \ldots,$ there exists a τ-regular p.m. λ_n on E such that $\mu = {}^*\lambda_n^n$, λ_n convoluted with itself n-times. μ is called *stable* if for each $a, b \in R$, a > 0, b > 0 there exist $c > 0$ and $x \in E$ such that $T_a\mu * T_b\mu = T_c\mu * \delta_x$, where (for a given $r \in R$, $r \neq 0$) $T_r\mu(A) = \mu(r^{-1}A)$ for every Borel set A, and δ_x denotes the degenerate p.m. at x.

Since τ-regularity is implicit in the definitions of i.d. and stable p.measures, in the sequel, whenever we talk of i.d. or stable p.m., it will be implicitly assumed that it is τ-regular. For more information on stable and i.d. p.measures see [3] and [9], respectively. A result from [3] which is relevant in the sequel is that for a given stable p.m. μ on a LCTVS E there exists a unique $\alpha \in (0, 2]$ such that c in the above definition is equal to $(a^\alpha + b^\alpha)^{1/\alpha}$; this α is referred to as the *index* of μ (note that if $\alpha = 2$, then μ is Gaussian, i.e. every continuous linear functional on E is a real Guassian (possibly degenerate) random variable).

DEFINITION 4. A p.m. μ on a separable normed linear space B is called *compound Poisson* if

$$\mu \equiv e(\nu) = e^{-\nu(B)} \sum_{k=0}^{\infty} \frac{{}^*\nu^k}{k!},$$

where ν is a finite measure and ${}^*\nu^0 = \delta_0$.

3. **Statement and proof of results.** We will prove the following two theorems:

THEOREM 1. *Let μ be a symmetric i.d. p.m. on a LCTVS E; then S_μ is a (closed) subgroup (under addition) of E.*

THEOREM 2. *Let μ be a symmetric stable p.m. of any index $\alpha \in (0, 2)$ on a LCTVS E; then S_μ is a (closed) subspace of E.*

For the proofs of these theorems we will need the following results.

LEMMA 1 [2, P. 37]. *Let μ and ν be two p. measures on a separable Banach space. Then $S_{\mu * \nu} = \overline{S_\mu + S_\nu}$.*

LEMMA 2 (TORTRAT [9]). *If μ is a symmetric i.d. p.m. on a separable Banach space B, then $\mu = \nu * \rho$, with ρ centered Gaussian and $\nu = \lim_n \nu_1 * \cdots * \nu_n$, where each ν_j is symmetric compound Poisson, and the limit is in the sense of* *weak convergence of p. measures. Further, for every* n, $\nu = \nu_1 * \cdots * \nu_n * \rho_n$, *for some symmetric i.d.p.m.* ρ_n.

PROOF. In view of Theorems II and V of [9], it remains to prove that the factors are symmetric and that ν is the limit of $\{\nu_1 * \cdots * \nu_n\}$ rather than being the limit of certain shifts of it. Since μ is symmetric i.d., $\mu = {}^*\mu_1^2$, where μ_1 is symmetric i.d. From the cited result of [9], $\mu_1 = \lambda * \sigma$, with σ Gaussian and $\lambda = \lim_n \lambda_1 * \cdots * \lambda_n * \delta_{x_n}$, where the λ_n's are compound Poisson and $x_n \in B$. Also

$$T_{-1}\mu_1 = T_{-1}\lambda * T_{-1}\sigma \quad \text{and} \quad T_{-1}\lambda = \lim_n T_{-1}\lambda_1 * \cdots * T_{-1}\lambda_n * \delta_{-x_n}.$$

Hence, by [3],

$$\nu \equiv \lambda * T_{-1}\lambda = \lim_n (\lambda_1 * T_{-1}\lambda_1) * \cdots * (\lambda_n * T_{-1}\lambda_n).$$

Hence taking $\nu_j = \lambda_j * T_{-1}\lambda_j$, $j = 1, 2, \ldots,$ and $\rho = \sigma * T_{-1}\sigma$, we get the first two conclusions of the lemma; the last conclusion also follows from similar arguments.

LEMMA 3. *Let* μ *be a symmetric i.d. p.m. on a separable Banach space* B. *Then* $0 \in S_\mu$.

PROOF. As in the proof above, $\mu = {}^*\mu_1^2$, where μ_1 is symmetric i.d. Hence, $S_{\mu_1} = -S_{\mu_1}$ and $S_\mu = \overline{S_{\mu_1} \pm S_{\mu_1}}$ (Lemma 1), implying $0 \in S_\mu$.

LEMMA 4. *Let* ν *be a finite measure on a separable Banach space* B *and* $\mu = e(\nu)$. *Then* $S_\mu = \overline{\bigcup_k kS_\nu}$, *where* kS_ν *is n-fold sum of* S_ν. *In particular,* S_μ *is a subsemigroup; further, if* μ *is symmetric then* S_μ *is a subgroup.*

PROOF. Since $S_{*\nu^k} = \overline{kS_\nu}$ (Lemma 1), $S_\mu \supseteq \overline{\bigcup_k kS_\nu}$. On the other hand,

$$\mu\left(\overline{\bigcup_k kS_\nu}\right) = \sum_k \frac{{}^*\nu^k(kS_\nu)e^{-\nu(E)}}{k!} = \sum_k \frac{{}^*\nu^k(E)e^{-\nu(E)}}{k!} = 1,$$

implying $\overline{\bigcup_k kS_\nu} \supseteq S_\mu$, completing the proof.

PROOF OF THEOREM 1. First we show that one needs to prove the result only in a separable Banach space. Let $\hat{E}$ be the completion of E and u be the natural injection of E into $\hat{E}$. Let $\nu = \mu \circ u^{-1}$; then one notes that ν is symmetric, τ-regular and i.d. Since $\hat{E}$ is a complete LCTVS, $\hat{E} =$ proj lim $g_{\alpha,\beta}E_\alpha$, the projective limit of a projective system $\{E_\alpha, g_{\alpha,\beta}, \alpha < \beta\}$ of Banach spaces relative to some index set I [8, p. 53]. For each $n = 1, 2, \ldots,$ let ν_n be a τ-regular p.m. on $\hat{E}$ such that $\nu = {}^*\nu_n^n$. Fix α, and let $\lambda_\alpha = \nu \circ \pi_\alpha^{-1}$, and $\lambda_{\alpha,n} = \nu_n \circ \pi_\alpha^{-1}$, $n = 1, 2, \ldots,$ where π_α is the natural projection of $\hat{E}$ into E_α. One notes that λ_α and $\lambda_{\alpha,n}$, $n = 1, 2, \ldots,$ are τ-regular; therefore, by [7, Lemma 2], S_{λ_α} and $S_{\lambda_{\alpha,n}}$, $n = 1, 2, \ldots,$ are separable subsets of E_α. Let B_α be the E_α-closure of the linear hull of the set $S_{\lambda_\alpha} \cup (\bigcup_{n=1}^\infty S_{\lambda_{\alpha,n}})$; then B_α is a separable Banach subspace of E_α. Further, since $\lambda_\alpha(B_\alpha) = \lambda_{\alpha,n}(B_\alpha) = 1$ and $\lambda_\alpha = {}^*\lambda_{\alpha,n}^n$ for all $n = 1, 2, \ldots, \lambda_\alpha$ restricted to B_α is i.d. and clearly symmetric. Therefore, if we can show that S_{λ_α} is a (closed) subgroup of B_α, then by [7, Proposition 1] S_ν would be a subgroup of E. Then since, by [7, Lemma 1], $S_\mu = S_\nu \cap E$, it will follow that S_μ is a subgroup.

Thus we assume that E is a separable Banach space and prove the result in the Banach space setting. By Lemma 2, $\mu = \nu * \rho$, where ρ is centered Gaussian and $\nu = \lim_n \nu_1 * \cdots * \nu_n$ with each ν_j symmetric compound Poisson. Thus, since S_ρ is known to be a subspace [6], [7] and, since by Lemma 1, $S_\mu = \overline{S_\nu + S_\rho}$, the proof will be complete if we can show that S_ν is a subgroup.

Let $\lambda_n = \nu_1 * \cdots * \nu_n$; then each λ_j is symmetric and compound Poisson. Therefore, by Lemma 4, S_{λ_n} is a subgroup. Thus, since $S_{\lambda_n} \uparrow$ as $n \uparrow$ (Lemma 1, again), it follows that $S \equiv \bigcup_{n=1}^\infty S_{\lambda_n}$ and, hence, $\bar{S}$ is a subgroup. We will complete the proof by showing that $\bar{S} = S_\nu$. Since $\bar{S}$ is closed and $\lim_n \lambda_n = \nu$, $\nu(\bar{S}) \geqslant \overline{\lim}\, \lambda_n(\bar{S}) = \overline{\lim}\, \lambda_n(S_{\lambda_n}) = 1$. Thus $\bar{S} \supseteq S_\nu$. From the second part of Lemma 2, we have, for every n, $\nu = \lambda_n * \rho_n$, where ρ_n is a symmetric i.d. p.m.; therefore, since by Lemma 3, $0 \in S_{\rho_n}$, we have, by Lemma 1, $S_\nu \supseteq S_{\lambda_n}$ for all n. Consequently, $S_\nu \supseteq \bar{S}$, completing the proof.

Proof of Theorem 2. This follows from Theorem 1, the fact that stable p. measures on a LCTVS are i.d. (Lemma 3.2 of [6]) and a result due to the author (see the proof of Theorem 3.1 of [6]), which asserts that for any symmetric stable p.m. μ on a LCTVS E, $aS_\mu \subseteq S_\mu$ for all real a with $|a| \leqslant 1$. We should point out, however, that even though in the statement of Theorem 3.1 of [6] it is assumed that $1 \leqslant \alpha \leqslant 2$, a careful look at the proof of this theorem given in [6] shows that this hypothesis is not needed to conclude that $aS_\mu \subseteq S_\mu$ for $|a| \leqslant 1$.

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