ON THE SUPPORT OF SYMMETRIC INFINITELY DIVISIBLE AND STABLE PROBABILITY MEASURES ON LCTVS

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Abstract. It is shown that the topological support (supp.) of a \( \tau \)-regular, symmetric, infinitely divisible (resp. stable of any index \( \alpha \in (0, 2) \)) probability measure on a Hausdorff LCTVS \( E \) is a subgroup (resp. a subspace) of \( E \). The part regarding the support of a stable probability measure of this theorem completes a result of A. De-Acosta [Ann. of Probability 3 (1975), 865-875], who proved a similar result for \( \alpha \in (1, 2) \), and the author [Proc. Amer. Math. Soc. 63 (1977), 306-312], who proved it for \( \alpha \in [1, 2) \). Further, it provides a complete affirmative solution to the question, raised by J. Kuelbs and V. Mandrekar [Studia Math. 50 (1974), 149-162], of whether the supp. of a symmetric stable probability measure of index \( \alpha \in (0, 1] \) on a separable Hilbert space \( H \) is a subspace of \( H \).

1. Introduction. The main purpose of this paper is to prove that the topological support (supp.) of a symmetric \( \tau \)-regular infinitely divisible (i.d.) (respectively, stable of any index \( \alpha \in (0, 2) \)) probability measure (p.m.) on a locally convex topological vector space (LCTVS) \( E \) is a (closed) subgroup (respectively, a subspace) of \( E \) (Theorems 1, 2). The part regarding the support of a stable p.m. of this result completes a result of De-Acosta [4], who proved a similar result for \( \alpha \in (1, 2) \), and the author [6], who proved it for \( \alpha \in [1, 2) \). It also settles a question raised by Kuelbs and Mandrekar [5].

2. Preliminaries. All topological spaces considered are assumed Hausdorff and all measures are defined on the Borel \( \sigma \)-algebra (the \( \sigma \)-algebra generated by the class of open sets) of the topological space. All vector spaces considered are over the real field \( R \). If \( A \) is a subset of a topological space \( E \), then \( A \) will denote, throughout, the \( E \)-closure of \( A \).

Definition 1. Let \( \mu \) be a finite measure on a topological space \( E \). Then \( \mu \) is called \( \tau \)-regular if \( \mu(\bigcup O_\alpha) = \lim \mu(O_\alpha) \) for every increasing net \( \{O_\alpha\} \) of open sets of \( E \). The unique smallest closed set with full \( \mu \)-measure (if it exists) is called the supp. of \( \mu \). It is easy to show that whenever the supp. of a finite measure exists, then it is the set of those points of the space, every open neighborhood (nbd.) of which has positive \( \mu \)-measure. Also, if \( \mu \) is \( \tau \)-regular...
then the supp. of $\mu$ always exists. If $\mu$ denotes a finite measure on a topological space, then, throughout, $S_\mu$ will denote the supp. of $\mu$.

**Definition 2.** Let $\mu$ and $\nu$ be two $\tau$-regular finite measures on a LCTVS $E$; then, according to [1] and [2], the convolution of $\mu$ and $\nu$, denoted by $\mu \ast \nu$, is defined by

$$
(\mu \ast \nu)(A) = \int_E \mu(A - x)\nu(dx) = \int_E \nu(A - x)\mu(dx),
$$

for every Borel set $A$.

**Definition 3.** Let $\mu$ be a $\tau$-regular p.m. on a LCTVS $E$; then $\mu$ is called i.d. if for each $n = 1, 2, \ldots$, there exists a $\tau$-regular p.m. $\lambda_n$ on $E$ such that $\mu = \ast_{\lambda_n}$; $\lambda_n$ convoluted with itself $n$-times. $\mu$ is called stable if for each $a, b \in R$, $a > 0$, $b > 0$ there exist $c > 0$ and $x \in E$ such that $T_\mu \ast T_\nu = T_\mu \ast \delta_x$, where (for a given $r \in R$, $r \neq 0$) $T_\mu(A) = \mu(r^{-1}A)$ for every Borel set $A$, and $\delta_x$ denotes the degenerate p.m. at $x$.

Since $\tau$-regularity is implicit in the definitions of i.d. and stable p.m., in the sequel, whenever we talk of i.d. or stable p.m., it will be implicitly assumed that it is $\tau$-regular. For more information on stable and i.d. p.m., see [3] and [9], respectively. A result from [3] which is relevant in the sequel is that for a given stable p.m. $\mu$ on a LCTVS $E$ there exists a unique $\alpha \in (0, 2]$ such that $c$ in the above definition is equal to $(a^\alpha + b^\alpha)^{1/\alpha}$; this $\alpha$ is referred to as the *index* of $\mu$ (note that if $\alpha = 2$, then $\mu$ is Gaussian, i.e. every continuous linear functional on $E$ is a real Gaussian (possibly degenerate) random variable).

**Definition 4.** A p.m. $\mu$ on a separable normed linear space $B$ is called compound Poisson if

$$
\mu \equiv e(\nu) = e^{-\nu(B)} \sum_{k=0}^{\infty} \frac{\nu^k}{k!},
$$

where $\nu$ is a finite measure and $\ast \nu^0 = \delta_0$.

3. **Statement and proof of results.** We will prove the following two theorems:

**Theorem 1.** Let $\mu$ be a symmetric i.d. p.m. on a LCTVS $E$; then $S_\mu$ is a (closed) subgroup (under addition) of $E$.

**Theorem 2.** Let $\mu$ be a symmetric stable p.m. of any index $\alpha \in (0, 2)$ on a LCTVS $E$; then $S_\mu$ is a (closed) subspace of $E$.

For the proofs of these theorems we will need the following results.

**Lemma 1** [2, p. 37]. Let $\mu$ and $\nu$ be two p. measures on a separable Banach space. Then $S_{\mu \ast \nu} = S_\mu + S_\nu$.

**Lemma 2** (Tortrat [9]). If $\mu$ is a symmetric i.d. p.m. on a separable Banach space $B$, then $\mu = \nu \ast \rho$, with $\rho$ centered Gaussian and $\nu = \lim_n \nu_1 \ast \cdots \ast \nu_n$, where each $\nu_j$ is symmetric compound Poisson, and the limit is in the sense of

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weak convergence of p. measures. Further, for every \( n, \nu = \nu_1 \ast \cdots \ast \nu_n \ast \rho_n \),
for some symmetric i.d. p.m. \( \rho_n \).

**Proof.** In view of Theorems II and V of [9], it remains to prove that the
factors are symmetric and that \( \nu \) is the limit of \( \{ \nu_1 \ast \cdots \ast \nu_n \} \) rather than
being the limit of certain shifts of it. Since \( \mu \) is symmetric i.d., \( \mu = \ast \mu_i^2 \),
where \( \mu_i \) is symmetric i.d. From the cited result of [9], \( \mu_1 = \lambda \ast \sigma \), with \( \sigma \)
Gaussian and \( \lambda = \lim_n \lambda_1 \ast \cdots \ast \lambda_n \ast \delta_{x_n} \), where the \( \lambda_n \)'s are compound
Poisson and \( x_n \in B \). Also
\[
T_{-1} \mu_1 = T_{-1} \lambda \ast T_{-1} \sigma \quad \text{and} \quad T_{-1} \lambda = \lim_n T_{-1} \lambda_1 \ast \cdots \ast T_{-1} \lambda_n \ast \delta_{-x_n}.
\]
Hence, by [3],
\[
\nu \equiv \lambda \ast T_{-1} \lambda = \lim_n (\lambda_1 \ast T_{-1} \lambda_1) \ast \cdots \ast (\lambda_n \ast T_{-1} \lambda_n).
\]
Hence taking \( \nu_j = \lambda_j \ast T_{-1} \lambda_j, j = 1, 2, \ldots \), and \( \rho = \sigma \ast T_{-1} \sigma \), we get the
first two conclusions of the lemma; the last conclusion also follows from
similar arguments.

**Lemma 3.** Let \( \mu \) be a symmetric i.d. p.m. on a separable Banach space \( B \).
Then \( 0 \in S_\mu \).

**Proof.** As in the proof above, \( \mu = \ast \mu_i^2 \), where \( \mu_i \) is symmetric i.d. Hence,
\( S_\mu = - S_\mu \), and \( S_\mu = S_\mu \pm S_\mu \) (Lemma 1), implying \( 0 \in S_\mu \).

**Lemma 4.** Let \( \nu \) be a finite measure on a separable Banach space \( B \) and
\( \mu = \nu (\nu) \). Then \( S_\mu = \cup_k kS_\nu \), where \( kS_\nu \) is n-fold sum of \( S_\nu \). In particular, \( S_\mu \) is
a subsemigroup; further, if \( \mu \) is symmetric then \( S_\mu \) is a subgroup.

**Proof.** Since \( S_\nu, k = kS_\nu \) (Lemma 1), \( S_\mu \supseteq \cup_k kS_\nu \). On the other hand,
\[
\mu \left( \bigcup_k kS_\nu \right) = \sum_k \frac{\ast \nu^k (kS_\nu) e^{-r(\nu)}}{k!} = \sum_k \frac{\ast \nu^k (E) e^{-r(\nu)}}{k!} = 1,
\]
implying \( \cup_k kS_\nu \supseteq S_\nu \), completing the proof.

**Proof of Theorem 1.** First we show that one needs to prove the result only
in a separable Banach space. Let \( \hat{E} \) be the completion of \( E \) and \( u \) be the
natural injection of \( E \) into \( \hat{E} \). Let \( \nu = \mu \circ u^{-1} \); then one notes that \( \nu \) is
symmetric, \( \tau \)-regular and i.d. Since \( \hat{E} \) is a complete LCTVS, \( \hat{E} = \text{proj lim} \ E_{\alpha, \beta} E_{\alpha} \), the projective limit of a projective system \( \{ E_{\alpha, \beta}, \alpha < \beta \} \)
of Banach spaces relative to some index set \( I \) [8, p. 53]. For each \( n = 1, 2, \ldots, \) let \( \nu_n \) be a \( \tau \)-regular p.m. on \( \hat{E} \) such that \( \nu = \ast \nu_n^n \). Fix \( \alpha \), and let
\( \lambda_n = \nu \circ \pi_\alpha^{-1} \), and \( \lambda_{n, n} = \nu_n \circ \pi_\alpha^{-1} \), \( n = 1, 2, \ldots \), where \( \pi_\alpha \) is the natural
projection of \( \hat{E} \) into \( E_\alpha \). One notes that \( \lambda_\alpha \) and \( \lambda_{n, n}, n = 1, 2, \ldots \), are
\( \tau \)-regular; therefore, by [7, Lemma 2], \( S_\lambda \) and \( S_{\lambda_{n, n}} \), \( n = 1, 2, \ldots \), are
separable subsets of \( E_\alpha \). Let \( B_\alpha \) be the \( E_\alpha \)-closure of the linear hull of the set
\( S_\lambda \cup (\bigcup_{n=1}^\infty S_{\lambda_{n, n}}) \); then \( B_\alpha \) is a separable Banach subspace of \( E_\alpha \). Further,
since \( \lambda_\alpha (B_\alpha) = \lambda_{n, n} (B_\alpha) = 1 \) and \( \lambda_\alpha = \ast \lambda_{n, n} \) for all \( n = 1, 2, \ldots, \lambda_\alpha \) restricted
to \( B_\alpha \) is i.d. and clearly symmetric. Therefore, if we can show that \( S_\lambda \) is a
(closed) subgroup of $B_a$, then by \cite[Proposition 1]{7} $S_\mu$ would be a subgroup of $E$. Then since, by \cite[Lemma 1]{7}, $S_\mu = S_\mu \cap E$, it will follow that $S_\mu$ is a subgroup.

Thus we assume that $E$ is a separable Banach space and prove the result in the Banach space setting. By Lemma 2, $\mu = \nu \ast \rho$, where $\rho$ is centered Gaussian and $\nu = \lim_n \nu_1 \ast \cdots \ast \nu_n$ with each $\nu_j$ symmetric compound Poisson. Thus, since $S_\rho$ is known to be a subspace \cite{6,7} and, since by Lemma 1, $S_\mu = S_\nu + S_\mu$, the proof will be complete if we can show that $S_\mu$ is a subgroup.

Let $\lambda_n = \nu_1 \ast \cdots \ast \nu_n$, then each $\lambda_j$ is symmetric and compound Poisson. Therefore, by Lemma 4, $S_{\lambda_n}$ is a subgroup. Thus, since $S_{\lambda_n} \uparrow$ as $n \uparrow$ (Lemma 1, again), it follows that $S = \bigcup_{n=1}^{\infty} S_{\lambda_n}$ and, hence, $S$ is a subgroup. We will complete the proof by showing that $S = S_{\mu}$. Since $S$ is closed and $\lim_n \lambda_n = \nu$, $\nu(S) = \lim_n \lambda_n(S) = 1$. Thus $S \supseteq S_{\mu}$. From the second part of Lemma 2, we have, for every $n$, $\nu = \lambda_n \ast \rho_n$, where $\rho_n$ is a symmetric i.d. p.m.; therefore, since by Lemma 3, $0 \in S_{\mu}$, we have, by Lemma 1, $S_{\nu} \supseteq S_{\lambda_n}$ for all $n$. Consequently, $S_{\nu} \supseteq S$, completing the proof.

**PROOF OF THEOREM 2.** This follows from Theorem 1, the fact that stable p. measures on a LCTVS are i.d. (Lemma 3.2 of \cite{6}) and a result due to the author (see the proof of Theorem 3.1 of \cite{6}), which asserts that for any symmetric stable p.m. $\mu$ on a LCTVS $E$, $aS_{\mu} \subseteq S_{\mu}$ for all real $a$ with $|a| < 1$. We should point out, however, that even though in the statement of Theorem 3.1 of \cite{6} it is assumed that $1 < \alpha < 2$, a careful look at the proof of this theorem given in \cite{6} shows that this hypothesis is not needed to conclude that $aS_{\mu} \subseteq S_{\mu}$ for $|a| < 1$.

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