MINIMAL INJECTIVE RESOLUTIONS UNDER FLAT BASE CHANGE

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Abstract. For a flat morphism \( \varphi: A \to B \) of noetherian rings, the minimal injective resolution of the \( B \)-module \( M \otimes_A B \) is described in terms of the minimal injective resolution of the finitely generated \( A \)-module \( M \) and the minimal injective resolutions of the fibers of \( \varphi \).

Throughout this note \( A \) and \( B \) denote commutative noetherian rings with multiplicative identities. For an \( A \)-module \( M \), a prime ideal \( p \) in \( A \), and an integer \( i \) the (cardinal) number \( \mu^i_p(M) \) denotes the dimension of \( \text{Ext}^i_A(A/p, M_p) = \text{Ext}^i_{A_p}(k(p), M_p) \) considered as a vector-space over the field \( k(p) = A_p/pA_p \). Now let

\[
0 \to M \to I^0 \to I^1 \to \cdots \to I^i \to \cdots
\]

be a minimal injective resolution of \( M \). Then \( \mu^i_p(M) = \mu^0_p(I^i) \) and this is the number of copies of the injective hull \( E^p_A(A/p) \) of \( A/p \) in the decomposition of \( I^i \) into indecomposable injective modules. For this fact, and for other facts concerning the numbers \( \mu^i_p(M) \) and minimal injective resolutions, consult Bass' paper [1] (in particular, §2).

Now let \( \varphi: A \to B \) be a ringhomomorphism making \( B \) into a flat \( A \)-module, and let \( M \) be a finitely generated \( A \)-module. The main result of this note describes the minimal injective resolution of the \( B \)-module \( M \otimes_A B \) by expressing the number \( \mu^n_B(q, M \otimes_A B) \) (for each \( n \) and each prime ideal \( q \) in \( B \)) in terms of the numbers \( \mu^n_A(q \cap A, M) \) and \( \mu^0_C(qC, C) \) where \( C \) is the fiber of \( \varphi \) at \( q \cap A \). All the numbers here are finite, since \( M \) is finitely generated.

Theorem. Let \( \varphi: A \to B \) be a flat ringhomomorphism, let \( q \) be a prime ideal in \( B \), and let \( C \) denote the ring \( B/(q \cap A) \) with the prime ideal \( q' = qC \). Then for a finitely generated \( A \)-module \( M \) and for all numbers \( n \) there is an equality of numbers

\[
\frac{\mu^0_B(q, M \otimes_A B)}{\mu^0_B(qC, C) \mu^0_A(q \cap A, M)} = \sum_{p + q = n} \mu^0_C(q', C) \mu^0_A(q \cap A, M).
\]

This result will be proved below, but first we mention two immediate

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applications to the injective dimensions of the $A$-module $M$ and the $B$-module $M \otimes_A B$. These dimensions are here denoted by $\text{id}_A M$ and $\text{id}_B(M \otimes_A B)$ respectively (and they might be infinite).

**Corollary 1.** Let $A$ and $B$ be local rings and let $\varphi: A \to B$ be a flat local ring homomorphism. If $M$ is a finitely generated nonzero $A$-module, then

$$
\text{id}_B(M \otimes_A B) = \text{id}_A M + \text{id}_C C,
$$

where $C = B/mB$ and $m$ is the maximal ideal in $A$.

In particular, $\text{id}_B(M \otimes_A B)$ is finite if and only if $\text{id}_A M$ is finite and $C$ is a Gorenstein ring.

**Proof.** Corollary 1 follows directly from the Theorem since $\text{id}_A M = \sup\{i|\mu'_A(m, M) + 0\}$ (and similarly for $\text{id}_B(M \otimes_A B)$ and $\text{id}_C C$) (cf. [1, (3.2) Corollary]). □

**Remark 1.** The main result of [4] is a result in the same direction as Corollary 1. It states that if $\varphi: A \to B$ is a flat ring homomorphism and if $E$ is an injective $A$-module, then

$$
\text{id}_B(E \otimes_A B) = \sup\{\text{id}_F(p) F(p) | p \in \text{Ass}_AE\}
$$

where $F(p) = k(p) \otimes_A B$ is the fiber at $p$.

Suppose that $A$ is local with maximal ideal $m$ and let $\hat{A}$ denote the completion of $A$ (in the $m$-adic topology). If there exists a Gorenstein module over $A$ (that is, a finitely generated module $G$ with depth $G = \text{id} G < \infty$), then all the fibers of $A \to \hat{A}$ are Gorenstein rings (cf. [9, (2.8) Theorem] and consult [8] and [3] for facts about Gorenstein modules). The next result generalizes this to arbitrary finitely generated modules of finite injective dimension.

**Corollary 2.** Let $M$ be a finitely generated nonzero module over the local ring $A$. If $\text{id}_A M < \infty$, then the formal fiber $C = \hat{A} \otimes_A k(p)$ at $p$ is a Gorenstein ring for all $p$ in the support of $M$.

**Proof (of Corollary 2).** The local rings of $C$ are of the form $C_q$ where $q$ is a prime ideal in $\hat{A}$ lying over $p$. Now apply Corollary 1 to the homomorphism $A_q \to \hat{A}_q$ using the fact that the $\hat{A}_q$-module $M_q \otimes_{A_q} \hat{A}_q = \hat{M}_q$ is of finite injective dimension. □

**Example.** Ferrand and Raynaud have constructed a domain of dimension one such that the generic formal fiber $\hat{A} \otimes_A Q$ is not a Gorenstein ring ($Q$ is the field of fractions), cf. [2]. In particular, this ring $A$ has no Gorenstein modules, and Corollary 2 shows even more: If the finitely generated module $M$ over this ring $A$ is of finite injective dimension, then $M$ is artinian.

A third application of the Theorem can be found in [5]. Namely: If $M$ is a finitely generated $A$-module and $p \subseteq q$ are two prime ideals, then $\mu^i_A(p, M) < \mu^{i+1}_A(q, M)$ for all $i$, where $l = \dim(A_q/pA_q)$.

**Proof of the Theorem.** Since the $\mu^i$ is unchanged under localization we may place ourself in the following situation: $p$ and $q$ are maximal ideals in,
respectively, $A$ and $B$ such that $\varphi(p) \subseteq q$. We are then required to prove
\begin{equation}
\mu^q_B(a, M \otimes_A B) = \sum_{p + q = n} \mu^p_C(aC, C) \mu^q_M(p, M)
\end{equation}
for all $n$, when $C = B/pB$.

To prove this we shall for any $A$-module $M$ construct $C$-linear isomorphisms
\begin{equation}
\text{Ext}^p_C(l, M \otimes_A B) \cong \prod_{p + q = n} \text{Ext}^p_C(l, C) \otimes_k \text{Ext}^q_M(k, M)
\end{equation}
where $k = A/p$ and $l = B/a = C/aC$. Then for $M$ finitely generated we get the desired formula by counting dimensions over $l$.

During the proof we work in the category of complexes of modules over the different rings considered. Recall that a morphism $\alpha : X \to Y$ between complexes of modules over a ring $R$ is called a quasi-isomorphism if it induces isomorphisms $H^i(\alpha) : H^i(X) \cong H^i(Y)$ on the cohomology modules for all $i$. This property of a morphism is preserved by any of the functors
$$
\text{Hom}_R(P, -), \quad \text{Hom}_R(\cdot, I), \quad \text{and} \quad \cdot \otimes_R M,
$$
where $P$ is a bounded above complex of $R$-projective modules, $I$ is a bounded below complex of $R$-injective modules and $M$ is an $R$-flat module, cf. [6, Chapter I, Lemma 6.2, p. 64].

To construct the isomorphisms in (1) we choose a minimal $A$-injective resolution $I$ of $M$ and a quasi-isomorphism $I \otimes_A B \to J$ where $J$ is a complex of $B$-injective modules with $J^i = 0$ for $i < 0$, cf. [6, Chapter I, Lemma 4.6, p. 42]. Choose, furthermore, a $C$-projective resolution $P$ of $/$. Then we have quasi-isomorphisms
\begin{align*}
M & \to I, \quad I \otimes_A B \to J, \quad \text{and} \quad P \to I.
\end{align*}

The isomorphisms in (1) are now established by defining quasi-isomorphisms $\alpha^*$ and $\beta^*$
\begin{equation}
\begin{array}{ccc}
\text{Hom}_B(P, J) & \cong \text{Hom}_C(P, \text{Hom}_A(k, I) \otimes_k C) \\
\text{Hom}_B(P, J) & \cong \text{Hom}_C(P, \text{Hom}_B(C, J))
\end{array}
\end{equation}
and identifying the two sides of (1) with the cohomology of the two sides of (2). Here $\alpha^*$ is induced by the quasi-isomorphism $\alpha : P \to I$ and since $J$ is a complex of $B$-injective modules we see that $\alpha^*$ is a quasi-isomorphism. And $\beta^*$ is induced by the composite $C$-linear morphism $\beta$ defined by the commutative diagram
\begin{equation}
\begin{array}{ccc}
\text{Hom}_A(k, I) \otimes_k C & \xrightarrow{\beta} & \text{Hom}_B(C, J) \\
\text{Hom}_A(k, I) \otimes_A B & \to & \text{Hom}_B(k \otimes_A B, I \otimes_A B) \to \text{Hom}_B(k \otimes_A B, J).
\end{array}
\end{equation}
To prove that \( \beta \) is a quasi-isomorphism we choose a resolution \( F \) of \( k \) by finitely generated \( A \)-free modules. Then we have a quasi-isomorphism \( F \to k \) and a commutative diagram of complexes of \( B \)-modules

\[
\begin{array}{ccc}
\Hom_A(k, I) \otimes_A B & \longrightarrow & \Hom_B(k \otimes_A B, I \otimes_A B) \\
\gamma_1 & & \gamma_2 \\
\Hom_A(F, I) \otimes_A B & \longrightarrow & \Hom_B(F \otimes_A B, I \otimes_A B) \to \Hom_B(F \otimes_A B, J).
\end{array}
\]

The morphism \( \gamma_1 \) is a quasi-isomorphism since \( I \) consists of \( A \)-injective modules and \( B \) is \( A \)-flat. The morphism \( \kappa \) is an isomorphism since each \( F' \) is finitely generated \( A \)-free. The morphism \( \delta \) is a quasi-isomorphism since \( F \otimes_A B \) consists of \( B \)-free modules and \( I \otimes_A B \to J \) is a quasi-isomorphism. The morphism \( \gamma_2 \) is a quasi-isomorphism since \( J \) consists of \( B \)-injective modules and \( F \otimes_A B \to k \otimes_A B \) is a quasi-isomorphism by flatness of \( B \). As \( \beta \) is a quasi-isomorphism and \( P \) a complex of \( C \)-projective modules it follows that \( \beta_* \) is a quasi-isomorphism.

Passing to cohomology we see by flatness of \( B \) that \( M \otimes_A B \to I \otimes_A B \) is a quasi-isomorphism and, hence, the composite \( M \otimes_A B \to I \otimes_A B \to J \) is a quasi-isomorphism, i.e. \( J \) is a \( B \)-injective resolution of \( M \otimes_A B \). Consequently,

\[
H^n[\Hom_B(I, J)] = \Ext^n_B(I, M \otimes_A B).
\]

To compute the cohomology of \( \Hom_C(P, \Hom_A(k, I) \otimes_k C) \), we remark, that since \( I \) is a minimal resolution, the differentials in \( \Hom_A(k, I) \) are zero. Then this holds for \( \Hom_A(k, I) \otimes_k C \) as well and so the differentials in \( \Hom_C(P, \Hom_A(k, I) \otimes_k C) \) are induced from \( P \) only. And we get for the cohomology

\[
H^n[\Hom_C(P, \Hom_A(k, I) \otimes_k C)]
= \prod_{p+q=n} H^p[\Hom_C(P, \Hom_A(k, I^q) \otimes_k C)]
= \prod_{p+1=n} \Ext_C^p(I, \Hom_A(k, I^q) \otimes_k C)
= \prod_{p=q=n} \Ext_C^p(I, C) \otimes_k \Hom_A(k, I^q)
= \prod_{p+q=n} \Ext_C^p(I, C) \otimes_k \Ext_A^q(k, M).
\]

Remark 2. The proof of the quasi-isomorphisms in (2) works for any base change situation

\[
\begin{array}{ccc}
B & \to & C = B \otimes_A k \\
\downarrow & & \downarrow \\
A & \to & k
\end{array}
\]
where $A \to B$ is a flat homomorphism of noetherian rings and $A \to k$ is a (module-) finite ring homomorphism and $l$ is any finitely generated $C$-module. In the derived category $D(C)$ (2) becomes an isomorphism

$$\text{RHom}_B(l, M \otimes_A B) = \text{RHom}_C(l, \text{RHom}_A(k, M) \otimes_k C)$$

(in the notation of [6]). However, the calculation of the cohomology of the right-hand side of (2') made in the end of the proof requires some extra conditions on the ring $k$, such as being a field.

**Remark 3.** Corresponding to (2) we have also a spectral sequence:

$$E_2^{pq} = \text{Ext}^q_C(l, \text{Ext}^p(k, M) \otimes_k C) \Rightarrow \text{Ext}^{p+q}_A(l, M \otimes_A B).$$

In the situation described at the beginning of the proof of the Theorem this spectral sequence gives the inequality

$$\mu^q_B(a, M \otimes_A B) \leq \sum_{p+q=n} \mu^p_C(aC, C) \mu^q_C(p, M).$$

This inequality has been studied in some special cases by Paugam in [7]. The equality (0) shows that all the differentials $E_r^{pq} \to E_r^{p+r,q-r+1}$ are zero (for $r > 2$).

**Bibliography**

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