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NEW TOPOLOGICAL EXTENSION PROPERTIES

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ABSTRACT. A topological property $\mathcal{P}$ is called an extension property if $\mathcal{P}$ is closed-hereditary and productive. In this paper new topological extension properties are introduced, which to some extent fill up a gap between complete regularity and compactness.

All spaces considered here are Tychonoff. As generalizations of compactness, major topological properties have been investigated (e.g. realcompactness, topological completeness, etc.). Mostly these properties are closed-hereditary and productive. A topological property $\mathcal{P}$ is called an extension property if $\mathcal{P}$ is closed-hereditary and productive. Some details and numerous examples of extension properties can be found in **[6]**.

In this paper, as generalizations of compactness and first countability, we introduce new topological extension properties which fill a gap between complete regularity and compactness.

We review some notation: For a cardinal $\mathfrak{m}$ the cardinal successor of $\mathfrak{m}$ is denoted by $\mathfrak{m}^+$. For an ordinal α the ordinal successor of α is denoted by $\alpha + 1$. We regard each cardinal as its initial ordinal. For any ordinal γ, $W(\gamma)$ denotes the space of all ordinals less than γ with the usual interval topology.

1. **$P_z(\mathfrak{m})$-compactness and P_z-compactness.** A covering $\mathfrak{U}$ of a space X is called a partition if each pair of members in $\mathfrak{U}$ is disjoint. Throughout this paper each member of a partition is assumed to be nonempty. A partition $\mathfrak{U}$ is called a z-partition if each member of $\mathfrak{U}$ is a zero-set. For a space X the Stone-Čech compactification of X is denoted by βX.

DEFINITION 1. Let $\mathfrak{m}$ be an infinite cardinal. A space X is called *$P_z(\mathfrak{m})$-compact* if for any $y \in \beta X - X$ there is a z-partition $\mathfrak{U}$ of X such that $|\mathfrak{U}| < \mathfrak{m}$ and such that $y \notin \mathrm{cl}_{\beta X} Z$ for any $Z \in \mathfrak{U}$. A space X is called *P_z-compact* if X is $P_z(\mathfrak{m})$-compact for some cardinal $\mathfrak{m}$.

Obviously for infinite cardinals $\mathfrak{m}$ and $\mathfrak{n}$ such that $\mathfrak{m} \leqslant \mathfrak{n}$, $P_z(\mathfrak{m})$-compactness implies $P_z(\mathfrak{n})$-compactness. It is also obvious that a space X is P_z-compact if and only if for any $y \in \beta X - X$ there is a z-partition $\mathfrak{U}$ of X such that $y \notin \mathrm{cl}_{\beta X} Z$ for any $Z \in \mathfrak{U}$.

We will omit the proof of the following simple lemma.

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LEMMA 1. *For a space X the following are equivalent.*

(1) *X is $P_z(\mathfrak{m})$-compact.*

(2) *There exists a compactification cX of X with the property that for any $y \in cX - X$ there is a z-partition $\mathfrak{U}$ of X such that $|\mathfrak{U}| < \mathfrak{m}$ and such that $y \notin cl_{cX}Z$ for any $Z \in \mathfrak{U}$.*

THEOREM 1. (1) *For any infinite cardinal $\mathfrak{m}$, $P_z(\mathfrak{m})$-compactness is an extension property.*

(2) *P_z-compactness is an extension property.*

PROOF. (1) Let Y be a $P_z(\mathfrak{m})$-compact space and let X be a closed subspace of Y. Take $cl_{\beta Y}X$ as a compactification of X. Then obviously condition (2) of Lemma 1 is satisfied. Hence X is $P_z(\mathfrak{m})$-compact. For a product X of $P_z(\mathfrak{m})$-compact spaces $\{Y_\lambda : \lambda \in \Lambda\}$, if we take the compactification $cX = \Pi\{\beta Y_\lambda : \lambda \in \Lambda\}$ of X, then condition (2) of Lemma 1 is satisfied. Hence X is $P_z(\mathfrak{m})$-compact.

(2) P_z-compactness is obviously closed-hereditary. Now let $\{Y_\lambda : \lambda \in \Lambda\}$ be a set of P_z-compact spaces. Then for each $\lambda \in \Lambda$ there is a cardinal $\mathfrak{m}_\lambda$ such that Y_λ is $P_z(\mathfrak{m}_\lambda)$-compact. Since Λ is a set, there is a cardinal $\mathfrak{m}$ such that $\mathfrak{m} \geqslant \mathfrak{m}_\lambda$ for each $\lambda \in \Lambda$. Then each Y_λ is $P_z(\mathfrak{m})$-compact by the remark after Definition 1, so $\Pi\{Y_\lambda : \lambda \in \Lambda\}$ is $P_z(\mathfrak{m})$-compact by (1). Hence P_z-compactness is productive.

THEOREM 2. *The following are equivalent for a space X.*

(1) *X is compact.*

(2) *X is $P_z(\aleph_0)$-compact.*

PROOF. Compact spaces are obviously $P_z(\aleph_0)$-compact. The converse is also obvious since $\bigcup\{cl_{\beta X}Z : Z \in \mathfrak{U}\} = \beta X$ for any finite z-partition $\mathfrak{U}$ of X.

Next, we study a relation between $P_z(\mathfrak{m})$-compactness and realcompactness.

THEOREM 3. (1) *$P_z(\aleph_1)$-compact spaces are realcompact.*

(2) *Realcompact spaces are $P_z((2^{\aleph_0})^+)$-compact.*

PROOF. (1) Let X be a $P_z(\aleph_1)$-compact space. Then for any $y \in \beta X - X$ there is a countable z-partition $\mathfrak{U}$ of X such that $y \notin cl_{\beta X}Z$ for any $Z \in \mathfrak{U}$. Hence there is a zero-set Z_y of βX such that $y \in Z_y$ and $Z_y \cap X = \emptyset$. So X is realcompact.

(2) Let X be a realcompact space. Then for any $y \in \beta X - X$ there is a real-valued continuous function f on βX such that $f(y) = 0$ and $f(x) > 0$ for any $x \in X$. Now let $\mathfrak{U}_y = \{f^{-1}(r) \cap X : r \in \mathbf{R},\ f^{-1}(r) \cap X \neq \emptyset\}$. Then obviously $\mathfrak{U}_y$ is a z-partition of X such that $|\mathfrak{U}_y| < (2^{\aleph_0})^+$, and $y \notin cl_{\beta X}Z$ for any $Z \in \mathfrak{U}_y$.

EXAMPLE 1. The converse of (1) in Theorem 3 is not true. In fact, since the real line $\mathbf{R}$ with the usual interval topology cannot have an infinite countable z-partition [**4**, p. 195], $\mathbf{R}$ is not $P_z(\aleph_1)$-compact.

EXAMPLE 2. The converse of (2) in Theorem 3 is not true. Since each point

of $W(\omega_1)$ is a zero-set, where ω_1 is the first uncountable ordinal, $W(\omega_1)$ is $P_z(\aleph_2)$-compact and hence $P_z((2^{\aleph_0})^+)$-compact. But $W(\omega_1)$ is not realcompact.

The referee requests a result that for any uncountable cardinal $\mathfrak{m}$ there is a space X which is $P_z(\mathfrak{m})$-compact, but not $P_z(\mathfrak{n})$-compact for any $\mathfrak{n} < \mathfrak{m}$. The following partially answers this request.

THEOREM 4. *Let $\mathfrak{m}$ be an uncountable cardinal. If*

(a) $\mathfrak{m} = \mathfrak{k}^+$ *where $\mathfrak{k}$ is a regular cardinal, or*

(b) $\mathfrak{m}$ *is a limit cardinal,*

then there is a $P_z(\mathfrak{m})$-compact space which is not $P_z(\mathfrak{n})$-compact for any $\mathfrak{n} < \mathfrak{m}$.

PROOF. (a) It is obvious that $W(\omega_0)$ is a $P_z(\aleph_1)$-compact space which is not $P_z(\aleph_0)$-compact, where ω_0 is the first infinite ordinal. Now let $\mathfrak{k}$ be an uncountable regular cardinal, and let

$$V(\mathfrak{k}^+) = W(\mathfrak{k}) - \{\alpha \in W(\mathfrak{k}) \colon \mathrm{cf}(\alpha) > \omega_0, \alpha \text{ is a limit ordinal}\},$$

where $\mathrm{cf}(\alpha)$ is the cofinality of α. Then $V(\mathfrak{k}^+)$ is $P_z(\mathfrak{k}^+)$-compact since each point of $V(\mathfrak{k}^+)$ is a zero-set and $|V(\mathfrak{k}^+)| < \mathfrak{k}^+$. On the other hand, it is obvious that $\beta V(\mathfrak{k}^+) = W(\mathfrak{k} + 1)$. By the regularity of $\mathfrak{k}$, for any z-partition $\mathfrak{U}$ of $V(\mathfrak{k}^+)$ such that $|\mathfrak{U}| < \mathfrak{k}$, there is a member of Z of $\mathfrak{U}$ which is cofinal in $V(\mathfrak{k}^+)$. This shows that the point $\mathfrak{k}$ of $\beta V(\mathfrak{k}^+) - V(\mathfrak{k}^+)$ is contained in $\bigcup \{\mathrm{cl}_{\beta V(\mathfrak{k}^+)} Z \colon Z \in \mathfrak{U}\}$ for any z-partition $\mathfrak{U}$ of $V(\mathfrak{k}^+)$ such that $|\mathfrak{U}| < \mathfrak{k}$. That is, $V(\mathfrak{k}^+)$ is not $P_z(\mathfrak{k})$-compact and hence not $P_z(\mathfrak{n})$-compact for any $\mathfrak{n} < \mathfrak{k}^+$.

(b) Let $\mathfrak{m}$ be a limit cardinal. Then there is a transfinite sequence $\{\mathfrak{k}_\lambda \colon \lambda \in \Lambda\}$ of regular cardinals such that $\mathfrak{k}_\lambda < \mathfrak{m}$ for any $\lambda \in \Lambda$ and $\sup\{\mathfrak{k}_\lambda \colon \lambda \in \Lambda\} = \mathfrak{m}$. Now for each $\lambda \in \Lambda$ let $V(\mathfrak{k}_\lambda^+)$ be the space constructed in the proof of (a). Then the product space $\Pi\{V(\mathfrak{k}_\lambda^+) \colon \lambda \in \Lambda\}$ is obviously $P_z(\mathfrak{m})$-compact, but not $P_z(\mathfrak{n})$-compact for any $\mathfrak{n} < \mathfrak{m}$.

Let E be a space. Then a space X is called *E-compact* if X is homeomorphic to a closed subspace of the topological power $E^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$ [**1**].

COROLLARY. *There does not exist a space E such that P_z-compactness is equivalent to E-compactness.*

PROOF. By Theorem 4, for any P_z-compact space E there is a P_z-compact space X which cannot be homeomorphic to a closed subspace of the topological power $E^{\mathfrak{m}}$ for any cardinal $\mathfrak{m}$.

THEOREM 5. (1) *Let X be a closed subspace of a product of spaces of which each point is a G_δ-set. Then X is P_z-compact.*

(2) *Topologically complete spaces are P_z-compact.*

PROOF. (1) Since spaces of which each point is a G_δ-set are obviously P_z-compact, this statement follows from the fact that P_z-compactness is an extension property.

(2) Since topologically complete spaces are closed subspaces of products of

metrizable spaces, (2) is a special case of (1).

It seems natural to ask whether every space is P_z-compact. In the next section we will show the existence of a space which is not P_z-compact.

2. **P-realcompactness.** A point x of a space X is a P-point if any G_δ-set containing x is a neighborhood of x. A space X is a P-space if every point of X is a P-point. For a space X, $\mathrm{p}X$ denotes the space with the underlying set equal to the underlying set of X and with the topology generated by all G_δ-sets of X (see [3]).

DEFINITION 2. A space X is called P-realcompact if $\mathrm{p}X$ is realcompact.

LEMMA 2 (CF. R. E. WHEELER [5]). *Realcompact spaces are P-realcompact.*

The proof of Lemma 2 is essentially the same as that of Wheeler [5] which shows that if X is topologically complete, so is $\mathrm{p}X$. The following lemma is trivial.

LEMMA 3. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a set of spaces. Then* $\mathrm{p}\prod\{X_\lambda : \lambda \in \Lambda\} = \mathrm{p}\prod\{\mathrm{p}X_\lambda : \lambda \in \Lambda\}$.

THEOREM 6. *P-realcompactness is an extension property.*

PROOF. Let X be a closed subspace of a P-realcompact space Y. Then since for any G_δ-set G of X there is a G_δ-set G' of Y such that $G = G' \cap X$, $\mathrm{p}X$ is a closed subspace of $\mathrm{p}Y$. Hence $\mathrm{p}X$ is realcompact since $\mathrm{p}Y$ is realcompact. Next, let $\{X_\lambda : \lambda \in \Lambda\}$ be a set of P-realcompact spaces. Then $\prod\{\mathrm{p}X_\lambda : \lambda \in \Lambda\}$ is realcompact as a product of realcompact spaces. By Lemma 2, $\mathrm{p}\prod\{\mathrm{p}X_\lambda : \lambda \in \Lambda\}$ is realcompact. Hence $\mathrm{p}\prod\{X_\lambda : \lambda \in \Lambda\}$ is realcompact by Lemma 3.

THEOREM 7. *Let X be a nonmeasurable P_z-compact space. Then X is P-realcompact.*

PROOF. We can assume that X is not realcompact by Lemma 2. By the definition of P_z-compactness, for any $y \in \upsilon X - X$ there is a z-partition $\mathfrak{A}_y$ of X such that $\mathrm{cl}_{\upsilon X} Z \not\ni y$ for any $Z \in \mathfrak{A}_y$ where υX is the Hewitt realcompactification of X. Then since $\mathrm{cl}_{\upsilon X} Z$ is a zero-set of υX for any zero-set Z of X, $\{\mathrm{cl}_{\upsilon X} Z : Z \in \mathfrak{A}_y\}$ is a z-partition of $\bigcup\{\mathrm{cl}_{\upsilon X} Z : Z \in \mathfrak{A}_y\}$. Further $\mathrm{cl}_{\upsilon X} Z$ is realcompact for any $Z \in \mathfrak{A}_y$. Hence $\mathrm{p}\mathrm{cl}_{\upsilon X} Z$ is realcompact for any $Z \in \mathfrak{A}_y$ by Lemma 2. Then $\mathrm{p}\bigcup\{\mathrm{cl}_{\upsilon X} Z : Z \in \mathfrak{A}_y\}$ is a topological sum of realcompact spaces $\{\mathrm{p}\mathrm{cl}_{\upsilon X} Z : Z \in \mathfrak{A}_y\}$ since each set of the form $\mathrm{cl}_{\upsilon X} Z$, with Z a zero-set of X, is a zero-set of υX. Now $|\mathfrak{A}_y|$ is nonmeasurable, hence $\mathrm{p}\bigcup\{\mathrm{cl}_{\upsilon X} Z : Z \in \mathfrak{A}_y\}$ is realcompact. Since

$$X = \bigcap\left\{\bigcup\{\mathrm{cl}_{\upsilon X} Z : Z \in \mathfrak{A}_y\} : y \in \upsilon X - X\right\},$$

$$\mathrm{p}X = \bigcap\left\{\mathrm{p}\bigcup\{\mathrm{cl}_{\upsilon X} Z : Z \in \mathfrak{A}_y\} : y \in \upsilon X - X\right\}.$$

Then $\mathrm{p}X$ is realcompact as an intersection of realcompact spaces.

COROLLARY. *The following are equivalent.*

(1) *There is a P_z-compact space which is not P-realcompact.*

(2) *There is a measurable cardinal.*

PROOF. If there is a measurable cardinal, then there is a discrete space D which is not realcompact. D is a P_z-compact space which is not P-realcompact.

EXAMPLE 3. There is a nonmeasurable space which is not P-realcompact. Hence there is a space which is not P_z-compact by Theorem 7. In fact there is a nonmeasurable P-space which is not realcompact (see 9L of [**2**]).

REMARK. If pX is Lindelöf or discrete, then X is P_z-compact. However the author does not know whether or not there exists a P-realcompact space which is not P_z-compact.

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