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A PROPERTY OF FINITE p-GROUPS

SHOICHI KONDO

ABSTRACT. Let $R(G)$ denote the character ring of a finite group G and let Λ be a commutative ring with identity. In this paper we show that if $G \neq \{1\}$, then $\Lambda \otimes_Z R(G)$ has only one maximal ideal if and only if G is a p-group and Λ has only one maximal ideal $\mathfrak{m}$ such that $\Lambda/\mathfrak{m}$ is of characteristic p.

1. Let $R(G)$ be the character ring of a finite group G which is generated by complex irreducible characters χ of G over the ring Z of rational integers. Then $R(G)$ is a commutative Z-algebra and its identity is the principal character 1_G of G. The characters χ form a free Z-basis of $R(G)$.

Let Λ be a commutative ring with identity. We define $R_\Lambda(G) = \Lambda \otimes_Z R(G) = \Sigma_\chi \Lambda\chi$, in which Λ is embedded by $\Lambda 1_G$. Then $R_\Lambda(G)$ is a Λ-algebra with a Λ-free basis $\{\chi\}$. Let θ be a primitive $|G|$th root of 1 in the complex field, and let $A = \Lambda \otimes_Z Z[\theta] = \Lambda[\theta]$. Then $R_\Lambda(G)$ is regarded as a subring of the direct product A^G, the ring of all functions of G which take their values in A. We note that A^G is integral over Λ, hence also over $R_\Lambda(G)$. This implies that any prime ideal of $R_\Lambda(G)$ is of the form

$$M(x, \mathfrak{p}) = \{ f \in R_\Lambda(G) | f(x) \in \mathfrak{p} \}$$

for some $x \in G$ and some prime ideal $\mathfrak{p}$ of A (cf. §11.4 of **[6]**).

We can define an augmentation ε: $R_\Lambda(G) \to \Lambda$ by $\varepsilon(f) = f(1)$ for $f \in R_\Lambda(G)$, where 1 denotes the identity of G. Hence we see that $R_\Lambda(G)$ is Noetherian if and only if Λ is. Let $\mathfrak{m}$ be a maximal ideal of Λ. Then $\varepsilon^{-1}(\mathfrak{m})$ is a maximal ideal of $R_\Lambda(G)$, and $\varepsilon^{-1}(\mathfrak{m}) = \mathfrak{m} + \operatorname{Ker} \varepsilon$. It is clear that $\operatorname{Ker} \varepsilon$ is an ideal of $R_\Lambda(G)$ generated by $\{\chi - \chi(1)\}$.

We say that Λ is semilocal if Λ has only a finite number of maximal ideals. Inparticular, Λ is called a local ring if Λ has exactly one maximal ideal $\mathfrak{m}$, and the field $\Lambda/\mathfrak{m}$ is called the residue field of Λ. We note that $R_\Lambda(G)$ is semilocal if and only if Λ is. Indeed, if $\mathfrak{m}$ is a maximal ideal of Λ, then $R_{\Lambda/\mathfrak{m}}(G)$ is an Artin ring. Since $R_{\Lambda/\mathfrak{m}}(G)$ is isomorphic to $R_\Lambda(G)/\mathfrak{m}R_\Lambda(G)$, there exists only a finite number of maximal ideals of $R_\Lambda(G)$ lying over $\mathfrak{m}$. Hence if Λ is semilocal, then $R_\Lambda(G)$ is also. The converse follows from the fact that $R_\Lambda(G)$ is integral over Λ.

Now we shall prove

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THEOREM 1. *Suppose $G \neq \{1\}$. Then $R_\Lambda(G)$ is local if and only if G is a p-group and Λ is a local ring with residue field of characteristic p.*

REMARK. The same result is shown in [5] for group rings of abelian groups.

PROOF. First we assume that G is a p-group and Λ has only one maximal ideal $\mathfrak{m}$ such that $\operatorname{char}(\Lambda/\mathfrak{m}) = p$. Let $M = M(x, \mathfrak{p})$ be any maximal ideal of $R_\Lambda(G)$, where $x \in G$ and $\mathfrak{p}$ is a maximal ideal of A. Then $\mathfrak{p} \cap \Lambda = \mathfrak{m}$; hence $\mathfrak{m} \subseteq M$ and $pZ = \mathfrak{p} \cap Z$. Since the identity 1 is only one p-regular element of G, we see that

$$\chi(x) \equiv \chi(1) \pmod{\mathfrak{p}}$$

for all χ (cf. Exercise 2 of 10.3 in [6]). This implies that $\chi - \chi(1) \in M$, and so $\operatorname{Ker} \varepsilon \subseteq M$. Hence we have $M = \varepsilon^{-1}(\mathfrak{m})$, which shows that $\varepsilon^{-1}(\mathfrak{m})$ is the unique maximal ideal of $R_\Lambda(G)$.

Conversely, we assume that $R_\Lambda(G)$ is local. Using the augmentation ε we see that Λ is local. If $\mathfrak{m}$ is the maximal ideal of Λ, then $\varepsilon^{-1}(\mathfrak{m})$ is the maximal ideal of $R_\Lambda(G)$.

In order to prove that G is a p-group, we note first that the order $|G|$ of G is a nonunit in Λ. Indeed, if $|G|$ is a unit in Λ, then $|G|^{-1} r_G$ is an idempotent of $R_\Lambda(G)$, where r_G denotes the regular character of G. Since $R_\Lambda(G)$ is local, we have $|G|^{-1} r_G = 0$ or 1. This is contrary to $G \neq \{1\}$. Hence there exists a prime divisor p of $|G|$ such that p is a nonunit in Λ, and therefore $\Lambda/\mathfrak{m}$ is of characteristic p.

Now we shall prove that G is a p-group. Suppose on the contrary that it is not so. Then there exists an element $x \neq 1$ of G whose order is not a power of p. Let f be the character of G induced from the principal character 1_P of a p-Sylow subgroup P of G. Then $f(x) = 0$ and

$$f(1) = |G|/|P| \not\equiv 0 \pmod{p}.$$

Since f is a nonunit in $R_\Lambda(G)$, we have $f \in \varepsilon^{-1}(\mathfrak{m})$. This implies $f(1) \in \mathfrak{m} \cap Z = pZ$, a contradiction.

2. In [3], [4] Chwe and Neggers defined Steinitz rings and proved that Λ is a Steinitz ring if and only if Λ has only one maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}$ is T-nilpotent. Allen and Neggers [1] applied this result to the case of group rings. As an analogue of the theorem of Allen and Neggers we prove the following result.

THEOREM 2. *Suppose $G \neq \{1\}$. Then $R_\Lambda(G)$ is a Steinitz ring if and only if G is a p-group and Λ is a Steinitz ring whose characteristic is a power of p.*

PROOF. Suppose that $R_\Lambda(G)$ is a Steinitz ring. Using the augmentation ε we see that Λ is a Steinitz ring. Hence it follows from Theorem 1 that G is a p-group and $p \in \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of Λ. Since $\mathfrak{m}$ is T-nilpotent, we see that p is nilpotent, hence the characteristic of Λ is a power of p.

Conversely, suppose that G is a p-group and Λ is a Steinitz ring of $\operatorname{char} \Lambda = p^i$. Let $\mathfrak{m}$ be the maximal ideal of Λ. Since $\Lambda/\mathfrak{m}$ is of characteristic

p, it follows from Theorem 1 that $R_\Lambda(G)$ has only one maximal ideal $\varepsilon^{-1}(\mathfrak{m})$. We must prove that $\varepsilon^{-1}(\mathfrak{m})$ is T-nilpotent. Since $\varepsilon^{-1}(\mathfrak{m}) = \mathfrak{m} + \operatorname{Ker} \varepsilon$, and since $\mathfrak{m}$ is T-nilpotent, it suffices to show that Ker ε is nilpotent.

Let α be the augmentation $A = \Lambda[\theta] \to \Lambda$. Then $\alpha^{-1}(\mathfrak{m}) = \mathfrak{m} + \operatorname{Ker} \alpha$ is a maximal ideal of A and Ker α is a principal ideal of A generated by $1 - \theta$. Since $(1 - \theta)^{|G|} \equiv 0 \pmod{pA}$, and since char $\Lambda = p^i$, we see that $(1 - \theta)^{i|G|} = 0$, and so Ker α is nilpotent. Hence $\alpha^{-1}(\mathfrak{m})$ is T-nilpotent and its elements are nilpotent.

Now we have seen in the proof of Theorem 1 that $\chi(x) - \chi(1)$ are in $\alpha^{-1}(\mathfrak{m})$ for all $x \in G$ and all χ. Since G is finite, it follows that $\chi - \chi(1)$ are nilpotent elements of $R_\Lambda(G)$, hence Ker ε is a nilpotent ideal. This completes the proof.

In the case of Noetherian rings, the above result gives the following

COROLLARY. *Suppose $G \neq \{1\}$. Then $R_\Lambda(G)$ is an Artin local ring if and only if G is a p-group and Λ is an Artin local ring whose characteristic is a power of p.*

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DEPARTMENT OF MATHEMATICS, SCHOOL OF EDUCATION, WASEDA UNIVERSITY, TOKYO, JAPAN