A PROPERTY OF FINITE $p$-GROUPS

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Abstract. Let $R(G)$ denote the character ring of a finite group $G$ and let $\Lambda$ be a commutative ring with identity. In this paper we show that if $G \neq \{1\}$, then $\Lambda \otimes Z R(G)$ has only one maximal ideal if and only if $G$ is a $p$-group and $\Lambda$ has only one maximal ideal $m$ such that $\Lambda/m$ is of characteristic $p$.

1. Let $R(G)$ be the character ring of a finite group $G$ which is generated by complex irreducible characters $\chi$ of $G$ over the ring $Z$ of rational integers. Then $R(G)$ is a commutative $Z$-algebra and its identity is the principal character $1_G$ of $G$. The characters $\chi$ form a free $Z$-basis of $R(G)$.

Let $\Lambda$ be a commutative ring with identity. We define $R_\Lambda(G) = \Lambda \otimes Z R(G) = \sum \chi \Lambda \chi$, in which $\Lambda$ is embedded by $\Lambda 1_G$. Then $R_\Lambda(G)$ is a $\Lambda$-algebra with a $\Lambda$-free basis $\{\chi\}$. Let $\theta$ be a primitive $|G|$th root of 1 in the complex field, and let $A = \Lambda \otimes Z[\theta] = \Lambda[\theta]$. Then $R_\Lambda(G)$ is regarded as a subring of the direct product $A^G$, the ring of all functions of $G$ which take their values in $A$. We note that $A^G$ is integral over $\Lambda$, hence also over $R_\Lambda(G)$. This implies that any prime ideal of $R_\Lambda(G)$ is of the form

$$M(x, \mathfrak{p}) = \{ f \in R_\Lambda(G) | f(x) \in \mathfrak{p} \}$$

for some $x \in G$ and some prime ideal $\mathfrak{p}$ of $A$ (cf. §11.4 of [6]).

We can define an augmentation $\epsilon: R_\Lambda(G) \to \Lambda$ by $\epsilon(f) = f(1)$ for $f \in R_\Lambda(G)$, where 1 denotes the identity of $G$. Hence we see that $R_\Lambda(G)$ is Noetherian if and only if $\Lambda$ is. Let $m$ be a maximal ideal of $\Lambda$. Then $\epsilon^{-1}(m)$ is a maximal ideal of $R_\Lambda(G)$, and $\epsilon^{-1}(m) = m + \text{Ker } \epsilon$. It is clear that $\text{Ker } \epsilon$ is an ideal of $R_\Lambda(G)$ generated by $\{\chi - \chi(1)\}$.

We say that $\Lambda$ is semilocal if $\Lambda$ has only a finite number of maximal ideals. In particular, $\Lambda$ is called a local ring if $\Lambda$ has exactly one maximal ideal $m$, and the field $\Lambda/m$ is called the residue field of $\Lambda$. We note that $R_\Lambda(G)$ is semilocal if and only if $\Lambda$ is. Indeed, if $m$ is a maximal ideal of $\Lambda$, then $R_\Lambda/m(G)$ is an Artin ring. Since $R_\Lambda/m(G)$ is isomorphic to $R_\Lambda(G)/mR_\Lambda(G)$, there exists only a finite number of maximal ideals of $R_\Lambda(G)$ lying over $m$. Hence if $\Lambda$ is semilocal, then $R_\Lambda(G)$ is also. The converse follows from the fact that $R_\Lambda(G)$ is integral over $\Lambda$.

Now we shall prove

Received by the editors April 23, 1976.
Key words and phrases. Finite $p$-group, character ring, Steinitz ring.

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THEOREM 1. Suppose $G \neq \{1\}$. Then $R_\Lambda(G)$ is local if and only if $G$ is a $p$-group and $\Lambda$ is a local ring with residue field of characteristic $p$.

REMARK. The same result is shown in [5] for group rings of abelian groups.

Proof. First we assume that $G$ is a $p$-group and $\Lambda$ has only one maximal ideal $m$ such that $\text{char}(\Lambda/m) = p$. Let $M = M(x, p)$ be any maximal ideal of $R_\Lambda(G)$, where $x \in G$ and $p$ is a maximal ideal of $A$. Then $p \cap \Lambda = m$; hence $m \subseteq M$ and $pZ = p \cap Z$. Since the identity $1$ is only one $p$-regular element of $G$, we see that

$$\chi(x) \equiv \chi(1) \pmod{p}$$

for all $\chi$ (cf. Exercise 2 of 10.3 in [6]). This implies that $\chi - \chi(1) \in M$, and so $\text{Ker } \epsilon \subseteq M$. Hence we have $M = \epsilon^{-1}(m)$, which shows that $\epsilon^{-1}(m)$ is the unique maximal ideal of $R_\Lambda(G)$.

Conversely, we assume that $R_\Lambda(G)$ is local. Using the augmentation $\epsilon$ we see that $\Lambda$ is local. If $m$ is the maximal ideal of $\Lambda$, then $\epsilon^{-1}(m)$ is the maximal ideal of $R_\Lambda(G)$.

In order to prove that $G$ is a $p$-group, we note first that the order $|G|$ of $G$ is a nonunit in $\Lambda$. Indeed, if $|G|$ is a unit in $\Lambda$, then $|G| - r_G^\Lambda$ is an idempotent of $R_\Lambda(G)$, where $r_G^\Lambda$ denotes the regular character of $G$. Since $R_\Lambda(G)$ is local, we have $|G| - r_G^\Lambda = 0$ or $1$. This is contrary to $G \neq \{1\}$. Hence there exists a prime divisor $p$ of $|G|$ such that $p$ is a nonunit in $\Lambda$, and therefore $\Lambda/m$ is of characteristic $p$.

Now we shall prove that $G$ is a $p$-group. Suppose on the contrary that it is not so. Then there exists an element $x \neq 1$ of $G$ whose order is not a power of $p$. Let $f$ be the character of $G$ induced from the principal character $1_\rho$ of a $p$-Sylow subgroup $P$ of $G$. Then $f(x) = 0$ and

$$f(1) = |G|/|P| \equiv 0 \pmod{p}.$$

Since $f$ is a nonunit in $R_\Lambda(G)$, we have $f \in \epsilon^{-1}(m)$. This implies $f(1) \in m \cap Z = pZ$, a contradiction.

2. In [3], [4] Chwe and Neggers defined Steinitz rings and proved that $\Lambda$ is a Steinitz ring if and only if $\Lambda$ has only one maximal ideal $m$ such that $m$ is $T$-nilpotent. Allen and Neggers [1] applied this result to the case of group rings. As an analogue of the theorem of Allen and Neggers we prove the following result.

THEOREM 2. Suppose $G \neq \{1\}$. Then $R_\Lambda(G)$ is a Steinitz ring if and only if $G$ is a $p$-group and $\Lambda$ is a Steinitz ring whose characteristic is a power of $p$.

Proof. Suppose that $R_\Lambda(G)$ is a Steinitz ring. Using the augmentation $\epsilon$ we see that $\Lambda$ is a Steinitz ring. Hence it follows from Theorem 1 that $G$ is a $p$-group and $p \in m$, where $m$ is the maximal ideal of $\Lambda$. Since $m$ is $T$-nilpotent, we see that $p$ is nilpotent, hence the characteristic of $\Lambda$ is a power of $p$.

Conversely, suppose that $G$ is a $p$-group and $\Lambda$ is a Steinitz ring of $\text{char } \Lambda = p^\gamma$. Let $m$ be the maximal ideal of $\Lambda$. Since $\Lambda/m$ is of characteristic
p, it follows from Theorem 1 that $R_A(G)$ has only one maximal ideal $\varepsilon^{-1}(m)$. We must prove that $\varepsilon^{-1}(m)$ is $T$-nilpotent. Since $\varepsilon^{-1}(m) = m + \ker \varepsilon$, and since $m$ is $T$-nilpotent, it suffices to show that $\ker \varepsilon$ is nilpotent.

Let $\alpha$ be the augmentation $A = \Lambda[\theta] \rightarrow \Lambda$. Then $\alpha^{-1}(m) = m + \ker \alpha$ is a maximal ideal of $A$ and $\ker \alpha$ is a principal ideal of $A$ generated by $1 - \theta$. Since $(1 - \theta)^{|G|} \equiv 0 \pmod{pA}$, and since $\text{char } A = p'$, we see that $(1 - \theta)^{|G|} = 0$, and so $\ker \alpha$ is nilpotent. Hence $\alpha^{-1}(m)$ is $T$-nilpotent and its elements are nilpotent.

Now we have seen in the proof of Theorem 1 that $\chi(x) - \chi(1)$ are in $\alpha^{-1}(m)$ for all $x \in G$ and all $\chi$. Since $G$ is finite, it follows that $\chi - \chi(1)$ are nilpotent elements of $R_A(G)$, hence $\ker \varepsilon$ is a nilpotent ideal. This completes the proof.

In the case of Noetherian rings, the above result gives the following

**Corollary.** Suppose $G \neq \{1\}$. Then $R_A(G)$ is an Artin local ring if and only if $G$ is a $p$-group and $\Lambda$ is an Artin local ring whose characteristic is a power of $p$.

**References**


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