THE BANACH-STONE PROPERTY
AND THE WEAK BANACH-STONE PROPERTY
IN THREE-DIMENSIONAL SPACES

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Abstract. Let $X$ and $Y$ be compact Hausdorff spaces, $E$ a Banach space, and $C(X, E)$ the space of continuous functions on $X$ to $E$. $E$ has the weak Banach-Stone property if, whenever $C(X, E)$ and $C(Y, E)$ are isometric, then $X$ and $Y$ are homeomorphic. $E$ has the Banach-Stone property if the descriptive as well as the topological conclusions of the Banach-Stone theorem for scalar functions remain valid in the case of isometries of $C(X, E)$ onto $C(Y, E)$. These two properties were first studied by M. Jerison, and it was later shown that every space $E$ found by Jerison to have the weak Banach-Stone property actually has the Banach-Stone property, thus raising the question of whether the two properties are distinct. Here we characterize all three-dimensional spaces with the weak Banach-Stone property, and, in so doing, show the properties to be distinct.

Throughout this article $X$ and $Y$ will denote compact Hausdorff spaces, $E$ a Banach space, and $C(X, E)$ the space of continuous functions on $X$ to $E$. $\mathcal{B}(E)$ will denote the space of bounded operators on $E$, given its strong operator topology, and $C(X)$ the space of continuous functions on $X$ to the scalar field associated with $E$.

We will say that $E$ has the Banach-Stone property if, given any isometry $A$ of $C(X, E)$ onto $C(Y, E)$, there exists a homeomorphism $\tau$ of $Y$ onto $X$ and a continuous function $y \to \varphi_y$ from $Y$ into $\mathcal{B}(E)$ such that, for all $y \in Y$, $\varphi_y$ is an isometry of $E$ onto itself, and such that $(A(F))(y) = \varphi_y F(\tau(y))$ for $F \in C(X, E)$, $y \in Y$—i.e. if the Banach-Stone theorem for $C(X)$ can be completely generalized for $C(X, E)$. $E$ has the weak Banach-Stone property if the existence of an isometry $A$ of $C(X, E)$ onto $C(Y, E)$ implies that $X$ and $Y$ are homeomorphic.

The Banach-Stone and weak Banach-Stone properties were first studied in [4] by M. Jerison, who showed that every $E$ belonging to the family of strictly convex spaces has the former property, and that all spaces $E$ belonging to a larger family have the latter. The weak Banach-Stone property has also been investigated by K. Sundaresan, who showed in [6] that for every positive integer $n \geqslant 2$, the space $l_n^{\infty}$ fails to have the weak Banach-Stone property.

In [1] a complete characterization of all finite-dimensional Banach spaces...
with the Banach-Stone property was obtained, and in [2] that same characterization was shown to hold within the family of reflexive Banach spaces. We say that a Banach space $E$ splits if $E$ can be written as the direct sum of two nonzero subspaces $U$ and $V$, $E = U \oplus V$, and has norm given by $\|e\| = \max\{\|u\|, \|v\|\}$ for $e = u + v \in E$. The characterization obtained in [1] and [2] is that if $E$ is a reflexive Banach space, then $E$ has the Banach-Stone property if, and only if, $E$ does not split [1, Theorems 1 and 2] and [2, Theorem].

It has also been established, [1, p. 92 and Theorem 2, p. 97], that every Banach space $E$ shown by Jerison in [4] to have the weak Banach-Stone property actually has the Banach-Stone property. Thus the question arises whether or not the two properties are indeed distinct. If $E$ is two dimensional, then it is trivially true that $E$ splits if, and only if, $E$ is isometric to $l_2^\infty$, so that by the results of [1] and [6] the properties do coincide for two-dimensional Banach spaces $E$.

In this article we characterize all three-dimensional spaces with the weak Banach-Stone property. It is shown that the only real (resp. complex) three-dimensional Banach space which fails to have the weak Banach-Stone property is the real (resp. complex) space $l_3^\infty$. Now it is easy to find three-dimensional spaces which split, and yet are not isometric to $l_3^\infty$. (For example, let $U$ be a two-dimensional Hilbert space, let $V$ be the corresponding scalar field, and form $E = U \oplus V$ giving $E$ the max norm.) Since such spaces have the weak Banach-Stone property, but not the Banach-Stone property, it is a consequence of the result obtained here that the two properties are distinct.

In this article we make use of the concept of a $T$-set as introduced by S. B. Myers in [5]. If $E$ is any Banach space, a subset $T$ of $E$ is called a $T$-set if, whenever \{\(e_1, \ldots, e_n\)\} is any finite subset of $T$, then $\|\sum_{j=1}^{n} e_j\| = \sum_{j=1}^{n} \|e_j\|$, and $T$ is maximal with respect to this property. We also use I. Singer's characterization of $C(X, E)^*$ as the Banach space of all regular Borel vector measures $m$ on $X$ to $E^*$, with finite variation $|m|$, and norm given by $\|m\| = |m|(X)$, see [3, p. 387]. For $x \in X$, $\mu_x$ will denote the scalar measure which is the positive unit mass concentrated at $x$, and we note for future reference that if $\varphi \in E^*$, then $\varphi \cdot \mu_x \in C(X, E)^*$.

Elements of $E$ will be denoted by $e$, $u$ and $v$ and those of $E^*$ by $\varphi$ and $\psi$. The value of $\varphi$ at $e$ is denoted by $\langle e, \varphi \rangle$. If $E = U \oplus V$, and if we write an element $e \in E$ as $e = u + v$, it is always implicit that $u \in U$ and $v \in V$. We denote elements of $C(X, E)$ and those of $C(Y, E)$, respectively, by $F$ and $G$, while elements of $C(X)$ and $C(Y)$ are denoted, respectively, by $f$ and $g$. The norms in $E$ and $E^*$ will be denoted by $\| \cdot \|$, while norms in $C(X, E)$, $C(Y, E)$, $C(X)$ and $C(Y)$ are denoted by $\| \cdot \|_\infty$.

If $E \neq \{0\}$ is a Banach space, and $T$ is a $T$-set in $E$, then there exists an element $\varphi \in E^*$ with $\|\varphi\| = 1$ such that $\langle e, \varphi \rangle = \|e\|$ for $e \in T$, [1, Proposition 1]. Let $\mathcal{F} = \{\varphi \in E^*: \|\varphi\| = 1 \text{ and } \langle e, \varphi \rangle = \|e\| \text{ for all } e \text{ belonging to } T\}$. This set is a vector space, and $\text{dim} \mathcal{F} = \text{dim} E = \text{dim} E^*$.
some $T$-set $T \subseteq E \}$. If $E$ is finite dimensional, then by a $T$-basis for $E^*$ we mean a basis consisting of elements of $\mathcal{F}$. Such bases always exist, [1, Proposition 2]. The following lemma is easily established.

**Lemma 1.** Let $E$ be a Banach space that splits, $E = U \oplus V$. Then $U^*$ is isometrically isomorphic to $V^0$, the annihilator of $V$ in $E^*$, under the map which sends an element $\varphi \in U^*$ to that element $\varphi' \in V^0 \subseteq E^*$ defined by $\langle e, \varphi' \rangle = \langle u, \varphi \rangle$, for $e = u + v \in E$.

Henceforth, we will cease to distinguish, in notation, between an element of $U^*$ and its image under the isometry of the lemma. The same symbol $\varphi$ may denote both an element of $U^*$ and its image in $V^0$ under the above correspondence.

**Lemma 2.** Let $E$ be a Banach space that splits, $E = U \oplus V$. Then given a $T$-set $T$ in $U$, the set $T = \{ u + v \in E: u \in T$ and $v$ is an element of $V$ with $\|v\| < \|u\| \}$ is a $T$-set in $E$.

**Proof.** One readily verifies that norm is an additive function on finite subsets of $T$. We show that $T$ is maximal in $E$ with respect to this property.

If $T$ were not maximal, there would exist an element $e_0 = u_0 + v_0 \in E - T$ such that $\|e_0 + e\| = \|e_0\| + \|e\|$ for all $e \in T$. Since $e_0 \not\in T$, either (i) $\|v_0\| > \|u_0\|$, or (ii) $\|v_0\| < \|u_0\|$ but $u_0 \not\in T$.

If (i) holds, choose $u \in T$ such that $\|u\| = \|v_0\| - \|u_0\|$. (This is possible, since $T$ is a cone, [5, Lemma 2.1, p. 133].) Then $u \in T$ and $\|e_0 + u\| = \|v_0\| < \|v_0\| + \|u\| = \|e_0\| + \|u\|$, contradicting the fact that $e_0$ must add in norm with every element of $T$.

If (ii) holds, again by [5, Lemma 2.1] there exists an element $u \in T$ such that $\|u_0 + u\| < \|u_0\| + \|u\|$. Then $u \in T$. We have

$$\|e_0\| + \|u\| = \|u_0\| + \|u\| > \|v_0\| + \|u\|,$$

and

$$\|e_0\| + \|u\| = \|u_0\| + \|u\| > \|v_0\| + \|u\| > \|v_0\|,$$

so that $\|e_0 + u\| = \max(\|u_0 + u\|, \|v_0\|) < \|e_0\| + \|u\|$, and we again reach a contradiction. The proof of the lemma is thus complete.

As an immediate consequence of Lemmas 1 and 2 we have the following:

**Lemma 3.** If $\varphi \in U^*$, $\|\varphi\| = 1$, and $\langle u, \varphi \rangle = \|u\|$ for all $u$ belonging to some $T$-set $T \subseteq U$, then considered as an element of $V^0 \subseteq E^*$, $\langle e, \varphi \rangle = \|e\|$ for all $e$ belonging to some $T$-set $T \subseteq E$.

**Lemma 4.** Let $E$ be a three-dimensional Banach space that splits, $E = U \oplus V$, where $U$ is two dimensional. Suppose that $U$ does not split, and let $\{ \varphi_1, \varphi_2 \}$ be a $T$-basis for $U^* = V^0 \subseteq E^*$. If $A$ is an isometry of $C(X, E)$ onto $C(Y, E)$ then for each $x \in X$ we have

$$A^{-1}(\varphi_i \cdot \mu_x) = \varphi_i' \cdot \mu_y, \quad i = 1, 2,$$
where the \( \varphi_i \) are elements of \( E^* \) with \( \| \varphi_i \| = 1 \), and \( y \) is an element of \( Y \) which depends only on \( x \).

**Proof.** By Lemma 3 and [1, Lemma 2, p. 94], for each \( x \in X \) and \( i = 1, 2 \), \( A^{-1}(\varphi_i \cdot \mu_x) \) is of the form \( \varphi'_i \cdot \mu_y \), where \( \| \varphi'_i \| = 1 \). We want to show that for fixed \( x \) we have \( y_1 = y_2 \).

Suppose, to the contrary, that for some \( x \in X \), we have \( y_1 \neq y_2 \). Then since \( U \) does not split, and \( \{ \varphi_1, \varphi_2 \} \) is a basis for \( U^* \), by [1, Lemma 1, p. 93] there exists a \( T \)-set \( T \) in \( U \) and a \( \varphi \in U^* = V^0 \) with \( \| \varphi \| = 1 \), such that \( \langle u, \varphi \rangle = \| u \| \) for all \( u \in T \), and such that both of the sets \( \{ \varphi, \varphi_1 \} \) and \( \{ \varphi, \varphi_2 \} \) are linearly independent.

By Lemma 3, \( \langle e, \varphi \rangle = \| e \| \) for all \( e \) belonging to some \( T \)-set \( T \) in \( E \), so that by [1, Lemma 2], \( A^*(\varphi \cdot \mu_x) \) is of the form \( \varphi' \cdot \mu_y \) for some \( \varphi' \in E^* \) with \( \| \varphi' \| = 1 \), and some \( y \in Y \). Let \( I \) be the subset of \( \{ 1, 2 \} \) such that for \( i \in I \), the support of \( A^{-1}(\varphi_i \cdot \mu_x) \) is equal to \( y \). Then \( I \) is either empty or a set containing one element.

We wish to show that \( \{ \varphi', \varphi'_i : i \in I \} \) is a linearly independent subset of \( E^* \). This is trivially true if \( I = \emptyset \), so suppose that \( I \) is a singleton, \( I = \{ i_0 \} \), and suppose that there exists a scalar \( \lambda \) such that \( \varphi' = \lambda \varphi'_i \). Then

\[
\varphi' \cdot \mu_y = \lambda \varphi'_i \cdot \mu_y = 0
\]

in \( C(Y, E)^* \), and hence

\[
A^*(\varphi' \cdot \mu_y - \lambda \varphi'_i \cdot \mu_y) = \varphi' \cdot \mu_x - \lambda \varphi'_i \cdot \mu_x = 0
\]

in \( C(X, E)^* \). Thus \( \varphi = \lambda \varphi'_i \in E^* \), contradicting the fact that \( \{ \varphi, \varphi'_i \} \) is a linearly independent set. Hence \( \{ \varphi', \varphi'_i : i \in I \} \) is linearly independent as claimed.

Thus we can take a vector \( e \in E \) such that \( \langle e, \varphi'_i \rangle = 0 \), \( i \in I \), but \( \langle e, \varphi' \rangle \neq 0 \). Let \( g \) be any element of \( C(Y) \) with \( g(y) \neq 0 \), and such that the support of \( g \) is disjoint from the support (or supports) of \( A^{-1}(\varphi_i \cdot \mu_x) \) for \( i \in I \), and define \( G \in C(Y, E) \) by \( G(y') = g(y') \cdot e \), \( y' \in Y \). Since \( \varphi \in U^* \) and \( \{ \varphi_1, \varphi_2 \} \) is a basis for \( U^* \), there exist scalars \( \lambda_i \) such that \( \varphi + \lambda_1 \varphi_1 + \lambda_2 \varphi_2 = 0 \). Thus \( \varphi \cdot \mu_x + \lambda_1 \varphi_1 \cdot \mu_x + \lambda_2 \varphi_2 \cdot \mu_x = 0 \) in \( C(X, E)^* \), and so

\[
0 = \int A^{-1}(G) d(\varphi \cdot \mu_x + \lambda_1 \varphi_1 \cdot \mu_x + \lambda_2 \varphi_2 \cdot \mu_x)
\]

\[
= \int Gd(A^{-1}(\varphi \cdot \mu_x + \lambda_1 \varphi_1 \cdot \mu_x + \lambda_2 \varphi_2 \cdot \mu_x))
\]

\[
= \int Gd(\varphi' \cdot \mu_y) + \sum_{i \in I} \lambda_i \int Gd(\varphi'_i \cdot \mu_y)
\]

\[
= \langle G(y), \varphi' \rangle + \sum_{i \in I} \lambda_i \langle G(y), \varphi'_i \rangle
\]

\[
= g(y) \langle e, \varphi' \rangle \neq 0.
\]

This contradiction completes the proof of the lemma.

**Theorem.** Let \( E \) be a three-dimensional Banach space. Then \( E \) fails to have
the weak Banach-Stone property if, and only if, \( E \) is isometric to \( l_3^{\infty} \).

**Proof.** The "if" part of the theorem has been established by Sundaresan, [6, p. 22]. Hence we assume that \( E \) is not isometric to \( l_3^{\infty} \) and show that \( E \) has the weak Banach-Stone property.

If \( E \) does not split, then \( E \) has the Banach-Stone property, and we are done. Thus suppose that \( E \) splits, \( E = U \oplus V \), where \( U \) is two dimensional and \( V \) is one dimensional. Now \( U \) cannot split, for otherwise \( E \) would be isometric to \( l_3^{\infty} \), contrary to the hypothesis.

Let \( A \) be an isometry of \( C(X, E) \) onto \( C(Y, E) \), and let \( \{ \varphi_1, \varphi_2 \} \) be a \( T \)-basis for \( U^* = V^0 \subseteq E^* \). Then for \( x \in X \), define \( \tau(x) = y \) if the supports of \( A^*^{-1}(\varphi_1 \cdot \mu_y) \) are equal to \( y \), for \( i = 1, 2 \). By Lemma 4, \( \tau \) is a well-defined function from \( X \) to \( Y \).

We wish to show, first of all, that \( \tau \) is one-one. Suppose, to the contrary, that \( x_1, x_2 \) are distinct points of \( X \), but that \( \tau(x_1) = \tau(x_2) = y \). This would mean that \( A^*^{-1}(\varphi_1 \cdot \mu_y) \) is of the form \( \varphi_j \cdot \mu_y \) for \( i = 1, 2 \) and \( j = 1, 2 \). Since \( E^* \) is three dimensional, \( \{ \varphi_j : i = 1, 2 \) and \( j = 1, 2 \} \) is a linearly dependent set, so that there exist scalars \( \alpha_j, \) not all zero, such that \( \sum_{i,j \in \{1, 2\}} \alpha_j \varphi_j = 0 \) in \( C(Y, E)^* \), so that

\[
A^* \left( \sum_{i,j \in \{1, 2\}} \alpha_j \varphi_j \right) = \sum_{i,j \in \{1, 2\}} \alpha_j \varphi_j \cdot \mu_y = 0
\]

in \( C(X, E)^* \). We may suppose, without loss of generality, that \( \alpha_{11} \neq 0 \). Since \( \varphi_1 \) and \( \varphi_2 \) are linearly independent, there exists an \( e \in E \) with \( \langle e, \varphi_2 \rangle = 0 \neq \langle e, \varphi_1 \rangle \). Take \( f \in C(X) \) with \( f(x_2) = 0 \neq f(x_1) \), and define \( F \in C(X, E) \) by \( F(x') = f(x') \cdot e \), for \( x' \in X \). We would then have

\[
0 = \int Fd \left( \sum_{i,j \in \{1, 2\}} \alpha_j \varphi_i \cdot \mu_x \right)
= \alpha_{11} \langle F(x_1), \varphi_1 \rangle + \alpha_{21} \langle F(x_1), \varphi_2 \rangle + 0
= \alpha_{11} f(x_1) \langle e, \varphi_1 \rangle \neq 0,
\]

which is absurd. Thus \( \tau \) is one-one, as claimed.

We next show that \( \tau \) maps \( X \) onto \( Y \). As above, we assume the contrary is true, and arrive at a contradiction. Suppose that \( y_1 \in Y - \tau(X) \). By Lemma 4 and [1, Lemma 2], with \( A^* \) replacing \( A^*^{-1} \), we would have \( A^*(\varphi_j \cdot \mu_y) \) is of the form \( \psi_j \cdot \mu_x, j = 1, 2 \), for some point \( x \in X \) dependent on \( y_1 \). Now \( \tau(x) = y_2 \), where by our assumption, \( y_2 \) cannot be equal to \( y_1 \). This means that \( A^*^{-1}(\varphi_1 \cdot \mu_y) = \psi_j \cdot \mu_y, i = 1, 2 \), where the \( \psi_j \) are elements of norm one in \( E^* \). Since \( \{ \varphi_1, \varphi_2, \psi_1, \psi_2 \} \) is a linearly dependent set in \( E^* \), there are scalars \( \lambda_i, \alpha_j, \) not all zero, such that \( \sum_{i,j \in \{1, 2\}} \lambda_i \varphi_i + \alpha_j \psi_j = 0 \), and thus

\[
\sum_{i,j \in \{1, 2\}} \lambda_i \varphi_i \cdot \mu_x + \alpha_j \psi_j \cdot \mu_x = 0
\]

in \( C(X, E)^* \). Hence
\[
A^{-1}\left( \sum_{i,j \in \{1,2\}} \lambda_i \varphi_i \cdot \mu_x + \alpha_j \psi_j \cdot \mu_x \right) \\
= \lambda_1 \varphi'_1 \cdot \mu_{y_2} + \lambda_2 \varphi'_2 \cdot \mu_{y_2} + \alpha_1 \varphi_1 \cdot \mu_{y_1} + \alpha_2 \varphi_2 \cdot \mu_{y_1} = 0
\]
in \(C(Y, E)^*\). If we note that since \(\varphi_1 \cdot \mu_x\) and \(\varphi_2 \cdot \mu_x\) are linearly independent elements of \(C(X, E)^*\), \(\varphi'_1 \cdot \mu_{y_2}\) and \(\varphi'_2 \cdot \mu_{y_2}\) are linearly independent in \(C(Y, E)^*\), and thus \(\varphi'_1\) and \(\varphi'_2\) are linearly independent in \(E^*\), a construction exactly analogous to that of the preceding paragraph then yields an element \(G \in C(Y, E)\) such that
\[
\int G d(\lambda_1 \varphi'_1 \cdot \mu_{y_2} + \lambda_2 \varphi'_2 \cdot \mu_{y_2} + \alpha_1 \varphi_1 \cdot \mu_{y_1} + \alpha_2 \varphi_2 \cdot \mu_{y_1}) \neq 0.
\]
This contradiction thus establishes that \(\tau\) is onto.

Finally, we show that \(\tau\) is continuous. Suppose, to the contrary, that there exists a net \(\{x_\beta : \beta \in B\}\) in \(X\) such that \(x_\beta \to x_0\), but that \(y_\beta = \tau(x_\beta) \neq \tau(x_0) = y_0\). Then there is a compact neighborhood \(N\) of \(y_0\) such that for every \(\beta_0 \in B\), there is a \(\beta > \beta_0\) such that \(y_\beta\) lies outside \(N\). Now by [1, Lemma 2] and by the definition of \(\tau\), \(A^{-1}(\varphi_1 \cdot \mu_{x_\beta}) = \varphi_{1,0} \cdot \mu_{y_\beta}\) for some \(\varphi_{1,0} \in E^*\) with \(\|\varphi_{1,0}\| = 1\). Fix \(e_0 \in E\) with \(\|e_0\| = 1\), such that \(\langle e_0, \varphi_{1,0} \rangle = 1\). Choose \(g_0 \in C(Y)\) with \(1 = \|g_0\| = g_0(y_0)\), and such that the support of \(g_0\) is contained in \(N\). Then define \(G_0 \in C(Y, E)\) by \(G_0(y) = g_0(y) \cdot e_0\), for \(y \in Y\). We have
\[
\langle (A^{-1}(G_0))(x_0), \varphi_1 \rangle = \int A^{-1}(G_0) d(\varphi_1 \cdot \mu_{x_\beta})
\]
\[
= \int G_0 d\left( A^{-1}(\varphi_1 \cdot \mu_{x_\beta}) \right) = \int G_0 d(\varphi_{1,0} \cdot \mu_{y_\beta})
\]
\[
= \langle G_0(y_0), \varphi_{1,0} \rangle = \langle e_0, \varphi_{1,0} \rangle = 1.
\]

Now since \(x_\beta \to x_0\) and \(\langle (A^{-1}(G_0))(x), \varphi_1 \rangle\) is a continuous function of \(x \in X\), there exists a \(\beta_0 \in B\) such that if \(\beta > \beta_0\) then \(\text{Re} \langle (A^{-1}(G_0))(x_\beta), \varphi_1 \rangle > \frac{1}{2}\). Thus fix a \(\beta > \beta_0\) such that \(y_\beta = \tau(x_\beta) \neq \tau(x_0) = y_0\). Again we have \(A^{-1}(\varphi_1 \cdot \mu_{x_\beta}) = \varphi_{1,\beta} \cdot \mu_{y_\beta}\) for some \(\varphi_{1,\beta} \in E^*\) with \(\|\varphi_{1,\beta}\| = 1\). Choose an \(e_\beta \in E\) with \(\|e_\beta\| = 1\) such that \(\langle e_\beta, \varphi_{1,\beta} \rangle = 1\). Then take an element \(g_\beta\) of \(C(Y)\) with \(1 = \|g_\beta\| = g_\beta(y_\beta)\), and such that the support of \(g_\beta\) is disjoint from \(N\). Define \(G_\beta \in C(Y, E)\) by \(G_\beta(y) = g_\beta(y) \cdot e_\beta\), for \(y \in Y\). Now \(G_0\) and \(G_\beta\) both have norm one and they have disjoint supports, so that \(\|G_0 + G_\beta\| = 1\). However,
\[
\|A^{-1}(G_0 + G_\beta)\| = \|A^{-1}(G_0) + A^{-1}(G_\beta)\| > \text{Re} \left[ \langle (A^{-1}(G_0))(x_\beta), \varphi_1 \rangle + \langle (A^{-1}(G_\beta))(x_\beta), \varphi_1 \rangle \right]
\]
\[
> \frac{1}{2} + \text{Re} \int A^{-1}(G_\beta) d(\varphi_1 \cdot \mu_{x_\beta})
\]
\[
= \frac{1}{2} + \text{Re} \int G_\beta d(\varphi_{1,\beta} \cdot \mu_{y_\beta}) = \frac{1}{2} + \text{Re} \int G_\beta d(\varphi_{1,\beta} \cdot \mu_{y_\beta})
\]
\[
= \frac{1}{2} + \text{Re} \langle G(y_\beta), \varphi_{1,\beta} \rangle = \frac{1}{2} + \text{Re} \langle e_\beta, \varphi_{1,\beta} \rangle = \frac{3}{2},
\]
which contradicts the fact that $A^{-1}$ is norm-preserving. Hence $\tau$ is a continuous, one-one map of $X$ onto $Y$, and is thus a homeomorphism.

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