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HOLOMORPHIC FOLIATIONS AND THE KOBAYASHI METRIC

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ABSTRACT. Here we extend the notion of the Kobayashi metric from complex manifolds to holomorphic foliations of smooth manifolds. This gives a notion of a hyperbolic foliation. We prove that a hyperbolic foliation of a compact manifold is compact and Hausdorff.

1. **Introduction.** Here we shall study foliations of C^∞ manifolds which admit some additional structure. In particular suppose M is a smooth ($= C^\infty$) manifold and $\mathfrak{F}$ is a foliation on M. We call $\mathfrak{F}$ compact if each leaf of the foliation $\mathfrak{F}$ is compact. Now $\mathfrak{F}$ defines an obvious equivalence relation on M by $x \sim y \Leftrightarrow x$ and y lie on the same leaf. We can therefore consider the topological space $M/\mathfrak{F}$ with the quotient topology defined by the equivalence relation $\sim$. A natural question to ask about the quotient space is–when can one assert that it is Hausdorff? We call a foliation $\mathfrak{F}$ Hausdorff if this is the case.

It is known that codimension-1 foliations with compact leaves are Hausdorff [3]. For codimension $\geqslant 2$ Reeb [8] has constructed examples of foliations with compact leaves which are not Hausdorff. Sullivan [9] has given an example of a codimension-4 foliation of $S^3 \times S^1 \times S^1$ with compact leaves which is not Hausdorff. There has, however, been much work aimed at giving conditions under which one can guarantee that a compact foliation is Hausdorff. In this direction we mention the work of Edwards, Millet, and Sullivan [3] and Epstein [4].

Here we assume that our foliation $\mathfrak{F}$ is holomorphic. This means that $\mathfrak{F}$ is given by a Haefliger cocycle with coefficients in the pseudogroup $\Gamma^q_{\mathbf{C}}$, of all local biholomorphic maps of $\mathbf{C}^q$. In order to conclude that a holomorphic foliation is Hausdorff we need some additional restrictions. To formulate these we introduce a notion which may be of some independent interest. We adapt the notion of the Kobayashi metric [7] defined for complex manifolds to the case of smooth manifolds with a holomorphic foliation. Thus we have a pseudometric canonically associated with a foliation. We call a holomorphic foliation hyperbolic if this pseudometric gives positive distance to points on distinct leaves of $\mathfrak{F}$. Our main result is the following theorem.

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THEOREM. *If $\mathfrak{F}$ is a holomorphic foliation on M, which is hyperbolic, then $\mathfrak{F}$ is Hausdorff. If M is compact then so is $\mathfrak{F}$.*

It is, perhaps, worth mentioning that if M is a compact hyperbolic complex manifold and $\mathfrak{F}$ is a holomorphic foliation that our metric is not the Kobayashi metric on M. Our metric measures distance only in directions normal to $\mathfrak{F}$. Hence if M is compact hyperbolic and $\mathfrak{F}$ is the foliation consisting of the single leaf M, our metric is trivial. If, however, $\mathfrak{F}$ is a compact hyperbolic holomorphic foliation on M it is not hard to show that $M/\mathfrak{F}$ is a complex V-manifold and that the metric defined here is the Kobayashi metric on $M/\mathfrak{F}$.

The plan of this paper is as follows: in §2 we will discuss holomorphic foliations, and prove a Newlander-Nirenberg Theorem for them. In §3, we will define the Kobayashi metric $d_{\mathfrak{F}}$ for a foliation and prove our main theorem and some immediate corollaries.

2. **Holomorphic foliations.** Let M be a manifold and $\mathfrak{F}$ a C^∞-, codimension-$2q$ foliation. Recall that $\mathfrak{F}$ can be presented as a Haefliger cocycle by choosing a cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and submersions $f_\alpha\colon U_\alpha \to \mathbf{R}^{2q}$ for each $\alpha \in \mathcal{A}$ so that the restriction of $\mathfrak{F}$ to U_α is given by the fibers of f_α. There are local C^∞-diffeomorphisms of $\mathbf{R}^{2q}$ for each pair $\alpha, \beta \in \mathcal{A}$ and $x \in U_\alpha \cap U_\beta$, labeled $g^x_{\alpha\beta}$ with the property that $f_\alpha = g^x_{\alpha\beta} \circ f_\beta$ near x and which satisfy the cocycle condition $g^x_{\alpha\gamma} = g^x_{\alpha\beta} \circ g^x_{\beta\gamma}$ for all $\alpha, \beta, \gamma \in \mathcal{A}$ and $x \in U_\alpha \cap U_\beta \cap U_\gamma$. We will use the notation $\{U_\alpha, f_\alpha, g^x_{\alpha\beta}\}$ for a Haefliger cocycle defining $\mathfrak{F}$ as above and we will call the maps $g^x_{\alpha\beta}$ transition maps.

DEFINITION. A foliation, $\mathfrak{F}$, presented as above is called holomorphic if the maps $g^x_{\alpha\beta}$ are all local biholomorphisms of $\mathbf{C}^q$.

In defining the Kobayashi metric it will be important to have a notion of holomorphic mappings of holomorphic foliations. For the general case of $(M_1, \mathfrak{F}_1)$ and $(M_2, \mathfrak{F}_2)$ such a notion may be difficult to define (it is not clear to us at the moment what the "right" definition should be). However for the case of $f\colon (M_1, \mathfrak{F}_1) \to (M_2, \mathfrak{F}_2)$ where M_1 is a complex manifold and $\mathfrak{F}_1$ is the foliation given by points, it is clear what the proper definition should be.

DEFINITION. Suppose V is a complex manifold and $\mathfrak{F}$ is a holomorphic foliation of the C^∞-manifold M. Then we say that a map $\varphi\colon V \to M$ is holomorphic with respect to $\mathfrak{F}$ if for all α the map $f_\alpha \circ \varphi\colon V \to \mathbf{C}^q$ is holomorphic.

REMARKS. (1) Since the f_α are holomorphically related it is clear that this notion is well defined. (2) An example of a holomorphic mapping is the following: since $\mathfrak{F}$ is a holomorphic foliation there are local varieties V transverse to the leaves of the foliation. The embeddings $\varphi\colon V \to M$ are holomorphic maps.

There is another natural way to characterize holomorphic foliations. Let Q be the normal bundle of the foliation and $B \to M$ the bundle of frames of Q.

DEFINITION. An almost complex structure on $\mathfrak{F}$ is a transverse $\mathrm{GL}(q, \mathbf{C})$

reduction of B. (See [1] for a definition of transverse G reduction.)

An almost complex structure on $\mathfrak{F}$ determines an endomorphism $J: Q \to Q$ with $J^2 = -I$. J induces a splitting of Q^* as $Q^* = Q^{*(1,0)} \oplus Q^{*(0,1)}$ in the usual way.

Let $\Omega^{\cdot}(Q^*)$ and $\Omega^{(p,q)}(Q^*)$ denote the spaces of sections of $\Lambda^{\cdot}(Q^*)$ and $\Lambda^{p,q}(Q^*)$ respectively. We include for the sake of the completeness, the Newlander-Nirenberg Theorem for foliations which is a special case of the complex Frobenius Theorem. Vaisman [10] has proven a similar theorem in the case where M is a complex manifold.

THEOREM 2.1. *Let $\mathfrak{F}$ be an almost complex foliation. Suppose $\mathfrak{F}$ satisfies the integrability condition $d\Omega^{(1,0)}(Q^*) \subseteq \Omega^{(1,0)}(Q^*) \wedge \Omega^1(M)$ is satisfied. Then $\mathfrak{F}$ is holomorphic.*

PROOF. Let $\mathfrak{F}$ be presented as a cocycle $\{U_\alpha, f_\alpha, g^x_{\alpha\beta}\}$. We will obtain a new presentation for $\mathfrak{F}$, $\{U_{\alpha'}, f'_{\alpha'}, g'^x_{\alpha'\beta'}\}$, in which the transition maps are local biholomorphisms.

It follows from [2, Definition 1.6(v) and Proposition 2.6] that the almost complex structure J induces unique almost complex structures J_α on the sets $V_\alpha = f(U_\alpha)$ and that the transition maps $g^x_{\alpha\beta}$ preserve these structures. The almost complex structure, J_α, is characterised by the fact that $f^*_\alpha(\Omega^{p,q}(V_\alpha)) \subseteq \Omega^{p,q}(Q^*|U_\alpha)$ for all $p, q \geqslant 0$.

We claim that the almost complex structures J_α are all integrable. To see this suppose that J_α is not integrable then by the classical Newlander-Nirenberg Theorem there is a 1-form $\phi \in \Omega^{(1,0)}(V_\alpha)$ with $d\phi = \tau + \varphi$ where $\varphi \in \Omega^{(0,2)}(V_\alpha)$ and $\tau \in \Omega^1(V_\alpha)$ and further $\varphi \neq 0$. But then $f^*_\alpha\phi \in \Omega^{(1,0)}(Q^*|V_\alpha)$ and $d(f^*_\alpha\phi) \notin \Omega^{(1,0)}(Q^*|V_\alpha) \wedge \Omega^1(Q^*)$ contradicting the integrability condition of the theorem.

By the above result it is clear that we can refine the cover $\{U_\alpha\}_{\alpha\in\mathcal{Q}}$ by a cover $\{U_{\alpha'}\}_{\alpha'\in\mathcal{Q}'}$ so that for $U_{\alpha'} \subseteq U_\alpha$ there is a biholomorphism, $h_{\alpha'}: f_\alpha(U'_{\alpha'}) \to \mathbf{C}^q$. Therefore, the cocycle defined by $f'_{\alpha'} = h_{\alpha'} \circ f_\alpha$ and $g'^x_{\alpha'\beta'} = h_{\beta'} \circ g^x_{\beta\alpha} \circ h^{-1}_{\alpha'}$ for $U_{\alpha'} \subseteq U_\alpha$ and $U_{\beta'} \subseteq U_\beta$ makes $\mathfrak{F}$ into a holomorphic foliation.

3. **The Kobayashi metric.** We now turn to a definition of the Kobayashi metric of a foliation. Our definition will be an adaptation of Kobayashi's definition for the case of a complex manifold.

Let $D_R = \{z \in \mathbf{C} \mid |z| < R\}$ be the disc of radius R in C, and let $p, q \in M$.

DEFINITION. (1) $d^*_{\mathfrak{F}}(p, q) = \inf \frac{1}{2} \log((R+1)/(R-1))$ where the infimum is taken over all $R > 1$ such that there is a holomorphic map $f: D_R \to (M, \mathfrak{F})$ with

$$f(0) = p, \qquad f(1) = q.$$

(2) $d_{\mathfrak{F}}(p, q) = \inf \sum_{i=0}^k d^*_{\mathfrak{F}}(p_i, p_{i+1})$ where the infimum is taken over all finite sequences $p = p_0, p_1, \ldots, p_k = q$.

REMARKS. This is Kobayashi's definition verbatim. What is different is our

notion of holomorphic mapping. As in the case of the usual Kobayashi metric, one easily verifies that

(i) $d_{\mathfrak{F}}(p, q) = 0$,

(ii) $d_{\mathfrak{F}}(p, q) = d_{\mathfrak{F}}(q, p)$,

(iii) $d_{\mathfrak{F}}(p_1, p_3) \leqslant d_{\mathfrak{F}}(p_1, p_2) + d_{\mathfrak{F}}(p_2, p_3)$,

(iv) $d_{\mathfrak{F}}$: $M \times M \to [0, \infty)$ is continuous.

Our next proposition gives the distance decreasing property which we shall require.

PROPOSITION 3.1. *Suppose V is a complex manifold and f: $V \to (M, \mathfrak{F})$ is holomorphic with respect to $\mathfrak{F}$. Let d_V denote the Kobayashi metric on V. Then $d_{\mathfrak{F}}(f(q), f(q)) \leqslant d_V(p, q)$.*

PROOF. This follows, trivially, from the definition of $d_{\mathfrak{F}}$ and d_V and the fact that if φ: $D_R \to V$ is holomorphic then so is $f \circ \varphi$.

PROPOSITION 3.2. *Suppose $(M, \mathfrak{F})$ is a holomorphic foliation. Suppose p, $q \in M$ are on the same leaf of $\mathfrak{F}$. Then $d_{\mathfrak{F}}(p, q) = 0$.*

PROOF. By Proposition 3.1, the fact that $d_{\mathbf{C}} \equiv 0$ and the triangle inequality it suffices to show that two points on the same leaf can be connected by a chain of holomorphic images of $\mathbf{C}$ in M. But any smooth map φ: $\mathbf{C} \to M$ whose image lies on a leaf of $\mathfrak{F}$ is holomorphic, since its projection to $\mathbf{C}$ under any f_α defining $\mathfrak{F}$ is a point.

The result now is easy: connect p to q by diffeomorphic copies of images of $\mathbf{C}$. This can always be done.

COROLLARY 3.3. *Let $(M, \mathfrak{F})$ be a holomorphic foliation and suppose that x, $x' \in L$ and y, $y' \in S$ where L and S are leaves of $\mathfrak{F}$. Then $d_{\mathfrak{F}}(x, y) = d_{\mathfrak{F}}(x', y')$.*

PROOF. We have the inequality $d_{\mathfrak{F}}(x, y) \leqslant d_{\mathfrak{F}}(x, x') + d_{\mathfrak{F}}(x', y') + d_{\mathfrak{F}}(y', y)$. But the last term is just $d_{\mathfrak{F}}(x', y')$. Similarly, $d_{\mathfrak{F}}(x', y') \leqslant d_{\mathfrak{F}}(x, y)$.

DEFINITION. A holomorphic foliation $(M, \mathfrak{F})$ is called hyperbolic if whenever x, y are on different leaves $d_{\mathfrak{F}}(x, y) > 0$.

In order to prove our theorem on hyperbolic foliations we will require the following result, Proposition 3.4 which follows from Lemma 3.5.

PROPOSITION 3.4. *If M is a compact manifold and $\mathfrak{F}$ is a foliation all of whose leaves are closed in M, then all leaves of $\mathfrak{F}$ are compact manifolds.*

LEMMA 3.5. *Let $\mathfrak{F}$ be a foliation of M and $N \xrightarrow{f} M$ a leaf. Suppose that $f(N)$ is a compact subset of M, then the manifold N is compact.*

PROOF. Suppose that N is not compact. Then there is an infinite sequence of points p_i of N having no limit point in N. Let $r_i = f(p_i)$. Since $f(N)$ is compact we may assume that $r_i \to r$ in $f(N)$. Pick an open set $U \subseteq M$ such that $r \in U$, and a chart g: $U \to \mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^q = \{(x, y) | x \in \mathbf{R}^p, y \in \mathbf{R}^q\}$ so that on U the leaves of $\mathfrak{F}$ are given by the planes $y =$ constant.

We may assume that an infinite number of the y-coordinates of the points r_i are distinct, for otherwise the sequence p_i would have a limit point in U.

Let $\mathbf{R}^q$ be identified with $\{0\} \times \mathbf{R}^q$ and let $X = f(N) \cap \mathbf{R}^q$. By the above remark X is infinite and it is closed since $f(N)$ is closed. We may assume that $r_i, r \in X$. Let r' be any other point of X. We claim that r' is an accumulation point of X. To see this, consider a path in $f(N)$ from r to r'. This induces a local homeomorphism h of $\mathbf{R}^q$ taking r to r'. Since $r_i \to r$ we have $h(r_i) \to h(r) = r'$ and $h(r_i) \in X$. Therefore every point of X is an accumulation point of X. An elementary argument shows that every closed subset of $\mathbf{R}^q$ with the property that every point is an accumulation point is uncountable. This means that $f(N) \cap U$ has an uncountable number of connected components since the components of $f(N) \cap U$ are in 1-1 correspondence with the points of X. Pulling back to N we see that N has an uncountable number of pairwise disjoint, open sets. But N is second countable. This contradiction proves the lemma.

LEMMA 3.6. *Let $\mathfrak{F}$ be a hyperbolic foliation on M. Then the leaf space $M/\mathfrak{F}$ is Hausdorff.*

PROOF. Let $p \in M$ and let $B(p, \varepsilon) = \{x \in M | d_{\mathfrak{F}}(p, x) < \varepsilon\}$. Since $d_{\mathfrak{F}}$ is continuous $B(p, \varepsilon)$ is open. Further, by Corollary 3.3, $B(p, \varepsilon)$ is a union of leaves of $\mathfrak{F}$. Let N_0 and N_1 be two leaves of $\mathfrak{F}$. To show that $M/\mathfrak{F}$ is Hausdorff we must find disjoint open sets O_0 and O_1 containing N_0 and N_1 respectively, and which are the union of leaves.

Let $\delta = d_{\mathfrak{F}}(N_0, N_1)$. Since $\mathfrak{F}$ is hyperbolic $\delta > 0$.

Let $p_0 \in N_0$, $p_1 \in N_1$. Then the required sets are $O_0 = B(p_0, \delta/2)$ and $O_1 = B(p_1, \delta/2)$.

Combining Proposition 3.4 and Lemma 3.6 we have the following.

THEOREM 3.7. *Every hyperbolic foliation of a compact manifold is compact and Hausdorff.*

REMARK. By Epstein [4] and Theorem 3.7 a hyperbolic foliation of a compact manifold is locally given by a Seifert fibration. From this it follows that the leaf space has the structure of a hyperbolic V manifold.

DEFINITION. A Hermitian foliation is a holomorphic foliation with a transverse $U(q)$-structure, compatible with the homomorphic structure.

PROPOSITION 3.8. *Let $\mathfrak{F}$ be a complex, codimension-q, hyperbolic foliation of a compact manifold M. Then $\mathfrak{F}$ is Hermitian.*

PROOF. Hamilton [5] has shown that every compact Hausdorff foliation is Riemannian. A Riemannian, holomorphic foliation is a $U(q)$-foliation.

PROPOSITION 3.9. *If $\mathfrak{F}$ is a hyperbolic foliation of a compact manifold M, then all secondary invariants defined by the characteristic homomorphism*

$$\Delta_*: H^{\cdot}\left(W(\mathrm{gl}(q, \mathbf{C}), U(q))_{2q}\right) \to H_{DR}(M, \mathbf{R})$$

are trivial.

PROOF. This is immediate from Proposition 3.9 and Theorem 4.52 of Kamber-Tondeur [6].

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