HOLOMORPHIC FOLIATIONS AND THE KOBAYASHI METRIC

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Abstract. Here we extend the notion of the Kobayashi metric from complex manifolds to holomorphic foliations of smooth manifolds. This gives a notion of a hyperbolic foliation. We prove that a hyperbolic foliation of a compact manifold is compact and Hausdorff.

1. Introduction. Here we shall study foliations of $C^\infty$ manifolds which admit some additional structure. In particular suppose $M$ is a smooth (= $C^\infty$) manifold and $\mathcal{F}$ is a foliation on $M$. We call $\mathcal{F}$ compact if each leaf of the foliation $\mathcal{F}$ is compact. Now $\mathcal{F}$ defines an obvious equivalence relation on $M$ by $x \sim y \iff x$ and $y$ lie on the same leaf. We can therefore consider the topological space $M/\mathcal{F}$ with the quotient topology defined by the equivalence relation $\sim$. A natural question to ask about the quotient space is—when can one assert that it is Hausdorff? We call a foliation $\mathcal{F}$ Hausdorff if this is the case.

It is known that codimension-1 foliations with compact leaves are Hausdorff [3]. For codimension $\geq 2$ Reeb [8] has constructed examples of foliations with compact leaves which are not Hausdorff. Sullivan [9] has given an example of a codimension-4 foliation of $S^3 \times S^1 \times S^1$ with compact leaves which is not Hausdorff. There has, however, been much work aimed at giving conditions under which one can guarantee that a compact foliation is Hausdorff. In this direction we mention the work of Edwards, Millet, and Sullivan [3] and Epstein [4].

Here we assume that our foliation $\mathcal{F}$ is holomorphic. This means that $\mathcal{F}$ is given by a Haefliger cocycle with coefficients in the pseudogroup $\Gamma_{qc}$ of all local biholomorphic maps of $C^\infty$. In order to conclude that a holomorphic foliation is Hausdorff we need some additional restrictions. To formulate these we introduce a notion which may be of some independent interest. We adapt the notion of the Kobayashi metric [7] defined for complex manifolds to the case of smooth manifolds with a holomorphic foliation. Thus we have a pseudometric canonically associated with a foliation. We call a holomorphic foliation hyperbolic if this pseudometric gives positive distance to points on distinct leaves of $\mathcal{F}$. Our main result is the following theorem.

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Theorem. If $\mathcal{F}$ is a holomorphic foliation on $M$, which is hyperbolic, then $\mathcal{F}$ is Hausdorff. If $M$ is compact then so is $\mathcal{F}$.

It is, perhaps, worth mentioning that if $M$ is a compact hyperbolic complex manifold and $\mathcal{F}$ is a holomorphic foliation that our metric is not the Kobayashi metric on $M$. Our metric measures distance only in directions normal to $\mathcal{F}$. Hence if $M$ is compact hyperbolic and $\mathcal{F}$ is the foliation consisting of the single leaf $M$, our metric is trivial. If, however, $\mathcal{F}$ is a compact hyperbolic holomorphic foliation on $M$ it is not hard to show that $M/\mathcal{F}$ is a complex $V$-manifold and that the metric defined here is the Kobayashi metric on $M/\mathcal{F}$.

The plan of this paper is as follows: in §2 we will discuss holomorphic foliations, and prove a Newlander-Nirenberg Theorem for them. In §3, we will define the Kobayashi metric $d_\mathcal{F}$ for a foliation and prove our main theorem and some immediate corollaries.

2. Holomorphic foliations. Let $M$ be a manifold and $\mathcal{F}$ a $C^\infty$, codimension-2 foliation. Recall that $\mathcal{F}$ can be presented as a Haefliger cocycle by choosing a cover $\{U_a\}_{a \in \mathcal{A}}$ of $M$ and submersions $f_a: U_a \to \mathbb{R}^{2q}$ for each $a \in \mathcal{A}$ so that the restriction of $\mathcal{F}$ to $U_a$ is given by the fibers of $f_a$. There are local $C^\infty$-diffeomorphisms of $\mathbb{R}^{2q}$ for each pair $a, \beta \in \mathcal{A}$ and $x \in U_a \cap U_\beta$, labeled $g_{a\beta}^x$ with the property that $f_a = g_{a\beta}^x \circ f_\beta$ near $x$ and which satisfy the cocycle condition $g_{a\gamma}^x = g_{a\beta}^x \circ g_{\beta\gamma}^x$ for all $a, \beta, \gamma \in \mathcal{A}$ and $x \in U_a \cap U_\beta \cap U_\gamma$. We will use the notation $(U_a, f_a, g_{a\beta}^x)$ for a Haefliger cocycle defining $\mathcal{F}$ as above and we will call the maps $g_{a\beta}^x$ transition maps.

Definition. A foliation, $\mathcal{F}$, presented as above is called holomorphic if the maps $g_{a\beta}^x$ are all local biholomorphisms of $\mathbb{C}^\nu$.

In defining the Kobayashi metric it will be important to have a notion of holomorphic mappings of holomorphic foliations. For the general case of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ such a notion may be difficult to define (it is not clear to us at the moment what the “right” definition should be). However for the case of $f: (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ where $M_1$ is a complex manifold and $\mathcal{F}_1$ is the foliation given by points, it is clear what the proper definition should be.

Definition. Suppose $V$ is a complex manifold and $\mathcal{F}$ is a holomorphic foliation of the $C^\infty$-manifold $M$. Then we say that a map $\varphi: V \to M$ is holomorphic with respect to $\mathcal{F}$ if for all $a$ the map $f_a \circ \varphi: V \to \mathbb{C}^\nu$ is holomorphic.

Remarks. (1) Since the $f_a$ are holomorphically related it is clear that this notion is well defined. (2) An example of a holomorphic mapping is the following: since $\mathcal{F}$ is a holomorphic foliation there are local varieties $V$ transverse to the leaves of the foliation. The embeddings $\varphi: V \to M$ are holomorphic maps.

There is another natural way to characterize holomorphic foliations. Let $Q$ be the normal bundle of the foliation and $B \to M$ the bundle of frames of $Q$.

Definition. An almost complex structure on $\mathcal{F}$ is a transverse $GL(q, \mathbb{C})$
reduction of $B$. (See [1] for a definition of transverse $G$ reduction.)

An almost complex structure on $\mathcal{F}$ determines an endomorphism $J: Q \to Q$ with $J^2 = -I$. $J$ induces a splitting of $\mathcal{Q}^*$ as $\mathcal{Q}^* = \mathcal{Q}^{*(1,0)} \oplus \mathcal{Q}^{*(0,1)}$ in the usual way.

Let $\Omega^p(\mathcal{Q}^*)$ and $\Lambda^p(\mathcal{Q}^*)$ denote the spaces of sections of $\Lambda^p(\mathcal{Q}^*)$ and $\Lambda^p(\mathcal{Q}^*)$ respectively. We include for the sake of the completeness, the Newlander-Nirenberg Theorem for foliations which is a special case of the complex Frobenius Theorem. Vaisman [10] has proven a similar theorem in the case where $M$ is a complex manifold.

**Theorem 2.1.** Let $\mathcal{F}$ be an almost complex foliation. Suppose $\mathcal{F}$ satisfies the integrability condition $d\Omega^{(1,0)}(\mathcal{Q}^*) \subseteq \Omega^{(1,0)}(\mathcal{Q}^*) \wedge \Omega^1(M)$ is satisfied. Then $\mathcal{F}$ is holomorphic.

**Proof.** Let $\mathcal{F}$ be presented as a cocycle $\{U_a, f_a, g_{aB}^x\}$. We will obtain a new presentation for $\mathcal{F}$, $\{U_a, f'_a, g'_{aB}^x\}$, in which the transition maps are local biholomorphisms.

It follows from [2, Definition 1.6(v) and Proposition 2.6] that the almost complex structure $J$ induces unique almost complex structures $J_a$ on the sets $V_a = f(U_a)$ and that the transition maps $g_{aB}^x$ preserve these structures. The almost complex structure, $J_a$, is characterised by the fact that $g_a^x(\mathcal{Q}^{(1,0)}(V_a)) \subseteq \mathcal{Q}^{p,q}(\mathcal{Q}^*|U_a)$ for all $p, q > 0$.

We claim that the almost complex structures $J_a$ are all integrable. To see this suppose that $J_a$ is not integrable then by the classical Newlander-Nirenberg Theorem there is a 1-form $\phi \in \Omega^{0,2}(V_a)$ with $d\phi = \tau + \varphi$ where $\varphi \in \Omega^{0,2}(V_a)$ and $\tau \in \Omega^1(V_a)$ and further $\varphi \neq 0$. But then $f_a^*\phi \in \Omega^{1,0}(Q^*|V_a)$ and $d(f_a^*\phi) \in \Omega^{1,0}(Q^*|V_a) \wedge \Omega^1(Q^*)$ contradicting the integrability condition of the theorem.

By the above result it is clear that we can refine the cover $\{U_a\}_{a \in \mathcal{A}}$ by a cover $\{U_{a'}\}_{a' \in \mathcal{A'}}$ so that for $U_{a'} \subseteq U_a$ there is a biholomorphism, $h_{a'}: f_a(U_{a'}) \to C$. Therefore, the cocycle defined by $f'_a = h_{a'} \circ f_a$ and $g'_{a'B} = h_{a'} \circ g_{aB}^x \circ h_{a'}^{-1}$ for $U_{a'} \subseteq U_a$ and $U_{a'} \subseteq U_{a'}$ makes $\mathcal{F}$ into a holomorphic foliation.

3. The Kobayashi metric. We now turn to a definition of the Kobayashi metric of a foliation. Our definition will be an adaptation of Kobayashi’s definition for the case of a complex manifold.

Let $D_R = \{ z \in \mathbb{C} | |z| < R \}$ be the disc of radius $R$ in $\mathbb{C}$, and let $p, q \in M$.

**Definition.** (1) $d_{\mathcal{F}}^s(p, q) = \inf \{ \frac{1}{2} \log((R + 1)/(R - 1)) \}$ where the infimum is taken over all $R > 1$ such that there is a holomorphic map $f: D_R \to (M, \mathcal{F})$ with $f(0) = p, \quad f(1) = q$.

(2) $d_{\mathcal{F}}(p, q) = \inf \sum_{i=0}^k d_{\mathcal{F}}^s(p_i, p_{i+1})$ where the infimum is taken over all finite sequences $p = p_0, p_1, \ldots, p_k = q$.

**Remarks.** This is Kobayashi’s definition verbatim. What is different is our
notion of holomorphic mapping. As in the case of the usual Kobayashi metric, one easily verifies that

(i) \( d_S(p, q) = 0 \),
(ii) \( d_S(p, q) = d_S(q, p) \),
(iii) \( d_S(p_1, p_3) \leq d_S(p_1, p_2) + d_S(p_2, p_3) \),
(iv) \( d_S: M \times M \to [0, \infty) \) is continuous.

Our next proposition gives the distance decreasing property which we shall require.

**Proposition 3.1.** Suppose \( V \) is a complex manifold and \( f: V \to (M, \mathcal{F}) \) is holomorphic with respect to \( \mathcal{F} \). Let \( d_V \) denote the Kobayashi metric on \( V \). Then \( d_S(f(q), f(p)) \leq d_V(p, q) \).

**Proof.** This follows, trivially, from the definition of \( d_S \) and \( d_V \) and the fact that if \( \varphi: D \to V \) is holomorphic then so is \( f \circ \varphi \).

**Proposition 3.2.** Suppose \( (M, \mathcal{F}) \) is a holomorphic foliation. Suppose \( p, q \in M \) are on the same leaf of \( \mathcal{F} \). Then \( d_S(p, q) = 0 \).

**Proof.** By Proposition 3.1, the fact that \( d_C \equiv 0 \) and the triangle inequality it suffices to show that two points on the same leaf can be connected by a chain of holomorphic images of \( \mathbb{C} \) in \( M \). But any smooth map \( \varphi: \mathbb{C} \to M \) whose image lies on a leaf of \( \mathcal{F} \) is holomorphic, since its projection to \( \mathbb{C} \) under any \( f_\alpha \) defining \( \mathcal{F} \) is a point.

The result now is easy: connect \( p \) to \( q \) by diffeomorphic copies of images of \( \mathbb{C} \). This can always be done.

**Corollary 3.3.** Let \( (M, \mathcal{F}) \) be a holomorphic foliation and suppose that \( x, x' \in L \) and \( y, y' \in S \) where \( L \) and \( S \) are leaves of \( \mathcal{F} \). Then \( d_S(x, y) = d_S(x', y') \).

**Proof.** We have the inequality \( d_S(x, y) \leq d_S(x, x') + d_S(x', y') + d_S(y', y) \). But the last term is just \( d_S(x', y') \). Similarly, \( d_S(x', y') \leq d_S(x, y) \).

**Definition.** A holomorphic foliation \( (M, \mathcal{F}) \) is called hyperbolic if whenever \( x, y \) are on different leaves \( d_S(x, y) > 0 \).

In order to prove our theorem on hyperbolic foliations we will require the following result, Proposition 3.4 which follows from Lemma 3.5.

**Proposition 3.4.** If \( M \) is a compact manifold and \( \mathcal{F} \) is a foliation all of whose leaves are closed in \( M \), then all leaves of \( \mathcal{F} \) are compact manifolds.

**Lemma 3.5.** Let \( \mathcal{F} \) be a foliation of \( M \) and \( N \to M \) a leaf. Suppose that \( f(N) \) is a compact subset of \( M \), then the manifold \( N \) is compact.

**Proof.** Suppose that \( N \) is not compact. Then there is an infinite sequence of points \( p_i \) of \( N \) having no limit point in \( N \). Let \( r_i = f(p_i) \). Since \( f(N) \) is compact we may assume that \( r_i \to r \) in \( f(N) \). Pick an open set \( U \subseteq M \) such that \( r \in U \), and a chart \( g: U \to \mathbb{R}^p = \mathbb{R}^p \times \mathbb{R}^q = \{(x, y)| x \in \mathbb{R}^p, y \in \mathbb{R}^q\} \) so that on \( U \) the leaves of \( \mathcal{F} \) are given by the planes \( y = \text{constant} \).
We may assume that an infinite number of the \( y \)-coordinates of the points \( r_i \) are distinct, for otherwise the sequence \( p_i \) would have a limit point in \( U \).

Let \( \mathbb{R}^d \) be identified with \( \{0\} \times \mathbb{R}^d \) and let \( X = f(N) \cap \mathbb{R}^d \). By the above remark \( X \) is infinite and it is closed since \( f(N) \) is closed. We may assume that \( r_i, r \in X \). Let \( r' \) be any other point of \( X \). We claim that \( r' \) is an accumulation point of \( X \). To see this, consider a path in \( f(N) \) from \( r \) to \( r' \). This induces a local homeomorphism \( h \) of \( \mathbb{R}^d \) taking \( r \) to \( r' \). Since \( r_i \to r \) we have \( h(r_i) \to h(r) = r' \) and \( h(r_i) \in X \). Therefore every point of \( X \) is an accumulation point of \( X \). An elementary argument shows that every closed subset of \( \mathbb{R}^d \) with the property that every point is an accumulation point is uncountable. This means that \( f(N) \cap U \) has an uncountable number of connected components since the components of \( f(N) \cap U \) are in 1-1 correspondence with the points of \( X \). Pulling back to \( N \) we see that \( N \) has an uncountable number of pairwise disjoint, open sets. But \( N \) is second countable. This contradiction proves the lemma.

**Lemma 3.6.** Let \( \mathcal{F} \) be a hyperbolic foliation on \( M \). Then the leaf space \( M/\mathcal{F} \) is Hausdorff.

**Proof.** Let \( p \in M \) and let \( B(p, \epsilon) = \{ x \in M | d_{\mathcal{F}}(p, x) < \epsilon \} \). Since \( d_{\mathcal{F}} \) is continuous \( B(p, \epsilon) \) is open. Further, by Corollary 3.3, \( B(p, \epsilon) \) is a union of leaves of \( \mathcal{F} \). Let \( N_0 \) and \( N_1 \) be two leaves of \( \mathcal{F} \). To show that \( M/\mathcal{F} \) is Hausdorff we must find disjoint open sets \( O_0 \) and \( O_1 \) containing \( N_0 \) and \( N_1 \) respectively, and which are the union of leaves.

Let \( \delta = d_{\mathcal{F}}(N_0, N_1) \). Since \( \mathcal{F} \) is hyperbolic \( \delta > 0 \).

Let \( p_0 \in N_0, p_1 \in N_1 \). Then the required sets are \( O_0 = B(p_0, \delta/2) \) and \( O_1 = B(p_1, \delta/2) \).

Combining Proposition 3.4 and Lemma 3.6 we have the following.

**Theorem 3.7.** Every hyperbolic foliation of a compact manifold is compact and Hausdorff.

**Remark.** By Epstein [4] and Theorem 3.7 a hyperbolic foliation of a compact manifold is locally given by a Seifert fibration. From this it follows that the leaf space has the structure of a hyperbolic \( V \) manifold.

**Definition.** A Hermitian foliation is a holomorphic foliation with a transverse \( (\mathcal{F}-) \)-structure, compatible with the homomorphic structure.

**Proposition 3.8.** Let \( \mathcal{F} \) be a complex, codimension-\( q \), hyperbolic foliation of a compact manifold \( M \). Then \( \mathcal{F} \) is Hermitian.

**Proof.** Hamilton [5] has shown that every compact Hausdorff foliation is Riemannian. A Riemannian, holomorphic foliation is a \( U(q) \)-foliation.

**Proposition 3.9.** If \( \mathcal{F} \) is a hyperbolic foliation of a compact manifold \( M \), then all secondary invariants defined by the characteristic homomorphism

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\[ \Delta_* : H' \left( W(\text{gl}(q, C), U(q)) \right)_{2q} \to H_{DR} (M, R) \]

are trivial.

**Proof.** This is immediate from Proposition 3.9 and Theorem 4.52 of Kamber-Tondeur [6].

**Bibliography**


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