HOLOMORPHIC FOLIATIONS AND THE KOBAYASHI METRIC

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ABSTRACT. Here we extend the notion of the Kobayashi metric from complex manifolds to holomorphic foliations of smooth manifolds. This gives a notion of a hyperbolic foliation. We prove that a hyperbolic foliation of a compact manifold is compact and Hausdorff.

1. Introduction. Here we shall study foliations of $\mathcal{C}^\infty$ manifolds which admit some additional structure. In particular suppose $M$ is a smooth (= $\mathcal{C}^\infty$) manifold and $\mathcal{F}$ is a foliation on $M$. We call $\mathcal{F}$ compact if each leaf of the foliation $\mathcal{F}$ is compact. Now $\mathcal{F}$ defines an obvious equivalence relation on $M$ by $x \sim y \iff x$ and $y$ lie on the same leaf. We can therefore consider the topological space $M/\mathcal{F}$ with the quotient topology defined by the equivalence relation $\sim$. A natural question to ask about the quotient space is—when can one assert that it is Hausdorff? We call a foliation $\mathcal{F}$ Hausdorff if this is the case.

It is known that codimension-1 foliations with compact leaves are Hausdorff [3]. For codimension $\geq 2$ Reeb [8] has constructed examples of foliations with compact leaves which are not Hausdorff. Sullivan [9] has given an example of a codimension-4 foliation of $S^3 \times S^1 \times S^1$ with compact leaves which is not Hausdorff. There has, however, been much work aimed at giving conditions under which one can guarantee that a compact foliation is Hausdorff. In this direction we mention the work of Edwards, Millet, and Sullivan [3] and Epstein [4].

Here we assume that our foliation $\mathcal{F}$ is holomorphic. This means that $\mathcal{F}$ is given by a Haefliger cocycle with coefficients in the pseudogroup $\mathbb{G}_{qc}$ of all local biholomorphic maps of $\mathcal{C}^\infty$. In order to conclude that a holomorphic foliation is Hausdorff we need some additional restrictions. To formulate these we introduce a notion which may be of some independent interest. We adapt the notion of the Kobayashi metric [7] defined for complex manifolds to the case of smooth manifolds with a holomorphic foliation. Thus we have a pseudometric canonically associated with a foliation. We call a holomorphic foliation hyperbolic if this pseudometric gives positive distance to points on distinct leaves of $\mathcal{F}$. Our main result is the following theorem.
THEOREM. If $\mathcal{F}$ is a holomorphic foliation on $M$, which is hyperbolic, then $\mathcal{F}$ is Hausdorff. If $M$ is compact then so is $\mathcal{F}$.

It is, perhaps, worth mentioning that if $M$ is a compact hyperbolic complex manifold and $\mathcal{F}$ is a holomorphic foliation that our metric is not the Kobayashi metric on $M$. Our metric measures distance only in directions normal to $\mathcal{F}$. Hence if $M$ is compact hyperbolic and $\mathcal{F}$ is the foliation consisting of the single leaf $M$, our metric is trivial. If, however, $\mathcal{F}$ is a compact hyperbolic holomorphic foliation on $M$ it is not hard to show that $M/\mathcal{F}$ is a complex $V$-manifold and that the metric defined here is the Kobayashi metric on $M/\mathcal{F}$.

The plan of this paper is as follows: in §2 we will discuss holomorphic foliations, and prove a Newlander-Nirenberg Theorem for them. In §3, we will define the Kobayashi metric $d_\mathcal{F}$ for a foliation and prove our main theorem and some immediate corollaries.

2. Holomorphic foliations. Let $M$ be a manifold and $\mathcal{F}$ a $C^\infty$, codimension-2 foliation. Recall that $\mathcal{F}$ can be presented as a Haefliger cocycle by choosing a cover $\{U_a\}_{a \in \mathcal{A}}$ of $M$ and submersions $f_a: U_a \to \mathbb{R}^q$ for each $a \in \mathcal{A}$ so that the restriction of $\mathcal{F}$ to $U_a$ is given by the fibers of $f_a$. There are local $C^\infty$-diffeomorphisms of $\mathbb{R}^q$ for each pair $a, \beta \in \mathcal{A}$ and $x \in U_a \cap U_\beta$, labeled $g^x_{a\beta}$ with the property that $f_a = g^x_{a\beta} \circ f_\beta$ near $x$ and which satisfy the cocycle condition $g^x_{a\gamma} = g^x_{a\beta} \circ g^x_{\beta\gamma}$ for all $a, \beta, \gamma \in \mathcal{A}$ and $x \in U_a \cap U_\beta \cap U_\gamma$. We will use the notation $\{U_a, f_a, g^x_{a\beta}\}$ for a Haefliger cocycle defining $\mathcal{F}$ as above and we will call the maps $g^x_{a\beta}$ transition maps.

Definition. A foliation, $\mathcal{F}$, presented as above is called holomorphic if the maps $g^x_{a\beta}$ are all local biholomorphisms of $\mathbb{C}^q$.

In defining the Kobayashi metric it will be important to have a notion of holomorphic mappings of holomorphic foliations. For the general case of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ such a notion may be difficult to define (it is not clear to us at the moment what the "right" definition should be). However for the case of $f: (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ where $M_1$ is a complex manifold and $\mathcal{F}_1$ is the foliation given by points, it is clear what the proper definition should be.

Definition. Suppose $V$ is a complex manifold and $\mathcal{F}$ is a holomorphic foliation of the $C^\infty$-manifold $M$. Then we say that a map $\varphi: V \to M$ is holomorphic with respect to $\mathcal{F}$ if for all $a$ the map $f_a \circ \varphi: V \to \mathbb{C}^q$ is holomorphic.

Remarks. (1) Since the $f_a$ are holomorphically related it is clear that this notion is well defined. (2) An example of a holomorphic mapping is the following: since $\mathcal{F}$ is a holomorphic foliation there are local varieties $V$ transverse to the leaves of the foliation. The embeddings $\varphi: V \to M$ are holomorphic maps.

There is another natural way to characterize holomorphic foliations. Let $Q$ be the normal bundle of the foliation and $B \to M$ the bundle of frames of $Q$.

Definition. An almost complex structure on $\mathcal{F}$ is a transverse $GL(q, \mathbb{C})$
reduction of $B$. (See [1] for a definition of transverse $G$ reduction.)

An almost complex structure on $\mathcal{F}$ determines an endomorphism $J: Q \to Q$ with $J^2 = -I$. $J$ induces a splitting of $Q^*$ as $Q^* = Q^{*(1,0)} \oplus Q^{*(0,1)}$ in the usual way.

Let $\Omega(Q^*)$ and $\Omega^{p,q}(Q^*)$ denote the spaces of sections of $\Lambda^p(Q^*)$ and $\Lambda^{p,q}(Q^*)$ respectively. We include for the sake of the completeness, the Newlander-Nirenberg Theorem for foliations which is a special case of the complex Frobenius Theorem. Vaisman [10] has proven a similar theorem in the case where $M$ is a complex manifold.

**Theorem 2.1.** Let $\mathcal{F}$ be an almost complex foliation. Suppose $\mathcal{F}$ satisfies the integrability condition $d\Omega^{1,0}(Q^*) \subseteq \Omega^{1,0}(Q^*) \wedge \Omega^1(M)$ is satisfied. Then $\mathcal{F}$ is holomorphic.

**Proof.** Let $\mathcal{F}$ be presented as a cocycle $\{U_a, f_a, g_b^a\}$. We will obtain a new presentation for $\mathcal{F}$, $\{U_a', f_a', g_b'^a\}$, in which the transition maps are local biholomorphisms.

It follows from [2, Definition 1.6(v) and Proposition 2.6] that the almost complex structure $J$ induces unique almost complex structures $J_a$ on the sets $V_a = f(U_a)$ and that the transition maps $g_b^a$ preserve these structures. The almost complex structure, $J_a$, is characterised by the fact that $f_a^*(\Omega^{p,q}(V_a)) \subseteq \Omega^{p,q}(Q^*|U_a)$ for all $p, q > 0$.

We claim that the almost complex structures $J_a$ are all integrable. To see this suppose that $J_a$ is not integrable then by the classical Newlander-Nirenberg Theorem there is a 1-form $\phi \in \Omega^{1,0}(V_a)$ with $d\phi = \tau + \phi$ where $\phi \in \Omega^{0,2}(V_a)$ and $\tau \in \Omega^1(V_a)$ and further $\phi \neq 0$. But then $f_a^*\phi \in \Omega^{1,0}(Q^*|V_a)$ and $d(f_a^*\phi) \notin \Omega^{1,0}(Q^*|V_a) \wedge \Omega^1(Q^*)$ contradicting the integrability condition of the theorem.

By the above result it is clear that we can refine the cover $\{U_a\}_{a \in \mathcal{A}}$ by a cover $\{U_{a'}\}_{a' \in \mathcal{A}'}$ so that for $U_{a'} \subseteq U_a$ there is a biholomorphism, $h_a: f_a(U_{a'}) \to \mathbb{C}^\nu$. Therefore, the cocycle defined by $f_{a'} = h_{a'} \circ f_a$ and $g_{b'}^a = h_{b'}^* \circ g_{a'}^b \circ h_{a'}^{-1}$ for $U_{a'} \subseteq U_a$ and $U_{b'} \subseteq U_b$ makes $\mathcal{F}$ into a holomorphic foliation.

3. **The Kobayashi metric.** We now turn to a definition of the Kobayashi metric of a foliation. Our definition will be an adaptation of Kobayashi's definition for the case of a complex manifold.

Let $D_R = \{z \in \mathbb{C} : |z| < R\}$ be the disc of radius $R$ in $\mathbb{C}$, and let $p, q \in M$.

**Definition.** (1) \[ d_{\mathbb{C}}(p, q) = \inf \frac{1}{2} \log((R + 1)/(R - 1)) \] where the infimum is taken over all $R > 1$ such that there is a holomorphic map $f: D_R \to (M, \mathcal{F})$ with $f(0) = p$, $f(1) = q$.

(2) \[ d_{\mathcal{F}}(p, q) = \inf \sum_{i=0}^{k} d_{\mathbb{C}}(p_i, p_{i+1}) \] where the infimum is taken over all finite sequences $p = p_0, p_1, \ldots, p_k = q$.

**Remarks.** This is Kobayashi's definition verbatim. What is different is our
notion of holomorphic mapping. As in the case of the usual Kobayashi metric, one easily verifies that

(i) \( d_{\mathcal{F}}(p, q) = 0 \),
(ii) \( d_{\mathcal{F}}(p, q) = d_{\mathcal{F}}(q, p) \),
(iii) \( d_{\mathcal{F}}(p_1, p_2) \leq d_{\mathcal{F}}(p_1, p_3) + d_{\mathcal{F}}(p_2, p_3) \),
(iv) \( d_{\mathcal{F}}: \mathcal{M} \times \mathcal{M} \to [0, \infty) \) is continuous.

Our next proposition gives the distance decreasing property which we shall require.

**Proposition 3.1.** Suppose \( \mathcal{V} \) is a complex manifold and \( f: \mathcal{V} \to (M, \mathcal{F}) \) is holomorphic with respect to \( \mathcal{F} \). Let \( d_{\mathcal{V}} \) denote the Kobayashi metric on \( \mathcal{V} \). Then
\[
\forall \xi \in \mathcal{V} d_{\mathcal{V}}(f(\xi), f(\eta)) \leq d_{\mathcal{V}}(\xi, \eta).
\]

**Proof.** This follows, trivially, from the definition of \( d_{\mathcal{F}} \) and \( d_{\mathcal{V}} \) and the fact that if \( \varphi: D \to \mathcal{V} \) is holomorphic then so is \( f \circ \varphi \).

**Proposition 3.2.** Suppose \( (M, \mathcal{F}) \) is a holomorphic foliation. Suppose \( p, q \in M \) are on the same leaf of \( \mathcal{F} \). Then \( d_{\mathcal{F}}(p, q) = 0 \).

**Proof.** By Proposition 3.1, the fact that \( d_{\mathcal{F}} \equiv 0 \) and the triangle inequality it suffices to show that two points on the same leaf can be connected by a chain of holomorphic images of \( \mathbb{C} \) in \( M \). But any smooth map \( \varphi: \mathbb{C} \to M \) whose image lies on a leaf of \( \mathcal{F} \) is holomorphic, since its projection to \( \mathbb{C} \) under any \( f_{\alpha} \) defining \( \mathcal{F} \) is a point.

The result now is easy: connect \( p \) to \( q \) by diffeomorphic copies of images of \( \mathbb{C} \). This can always be done.

**Corollary 3.3.** Let \( (M, \mathcal{F}) \) be a holomorphic foliation and suppose that \( x, x' \in L \) and \( y, y' \in S \) where \( L \) and \( S \) are leaves of \( \mathcal{F} \). Then \( d_{\mathcal{F}}(x, y) = d_{\mathcal{F}}(x', y') \).

**Proof.** We have the inequality \( d_{\mathcal{F}}(x, y) \leq d_{\mathcal{F}}(x, x') + d_{\mathcal{F}}(x', y') + d_{\mathcal{F}}(y, y') \). But the last term is just \( d_{\mathcal{F}}(x', y') \). Similarly, \( d_{\mathcal{F}}(x', y') \leq d_{\mathcal{F}}(x, y) \).

**Definition.** A holomorphic foliation \( (M, \mathcal{F}) \) is called hyperbolic if whenever \( x, y \) are on different leaves \( d_{\mathcal{F}}(x, y) > 0 \).

In order to prove our theorem on hyperbolic foliations we will require the following result, Proposition 3.4 which follows from Lemma 3.5.

**Proposition 3.4.** If \( M \) is a compact manifold and \( \mathcal{F} \) is a foliation all of whose leaves are closed in \( M \), then all leaves of \( \mathcal{F} \) are compact manifolds.

**Lemma 3.5.** Let \( \mathcal{F} \) be a foliation of \( M \) and \( N \to M \) a leaf. Suppose that \( f(N) \) is a compact subset of \( M \), then the manifold \( N \) is compact.

**Proof.** Suppose that \( N \) is not compact. Then there is an infinite sequence of points \( p_i \) of \( N \) having no limit point in \( N \). Let \( r_i = f(p_i) \). Since \( f(N) \) is compact we may assume that \( r_i \to r \) in \( f(N) \). Pick an open set \( U \subseteq M \) such that \( r \in U \), and a chart \( g: U \to \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q = \{(x, y) | x \in \mathbb{R}^p, y \in \mathbb{R}^q \} \) so that on \( U \) the leaves of \( \mathcal{F} \) are given by the planes \( y = \text{constant} \).
We may assume that an infinite number of the $y$-coordinates of the points $r_i$ are distinct, for otherwise the sequence $p_i$ would have a limit point in $U$.

Let $\mathbb{R}^s$ be identified with $\{0\} \times \mathbb{R}^s$ and let $X = f(N) \cap \mathbb{R}^s$. By the above remark $X$ is infinite and it is closed since $f(N)$ is closed. We may assume that $r_i, r \in X$. Let $r'$ be any other point of $X$. We claim that $r'$ is an accumulation point of $X$. To see this, consider a path in $f(N)$ from $r$ to $r'$. This induces a local homeomorphism $h$ of $\mathbb{R}^s$ taking $r$ to $r'$. Since $r \to r$ we have $h(r_i) \to h(r) = r'$ and $h(r_i) \in X$. Therefore every point of $X$ is an accumulation point of $X$. An elementary argument shows that every closed subset of $\mathbb{R}^s$ with the property that every point is an accumulation point is uncountable. This means that $f(N) \cap U$ has an uncountable number of connected components since the components of $f(N) \cap U$ are in 1-1 correspondence with the points of $X$. Pulling back to $N$ we see that $N$ has an uncountable number of pairwise disjoint, open sets. But $N$ is second countable. This contradiction proves the lemma.

Lemma 3.6. Let $\mathcal{F}$ be a hyperbolic foliation on $M$. Then the leaf space $M/\mathcal{F}$ is Hausdorff.

Proof. Let $p \in M$ and let $B(p, \varepsilon) = \{x \in M|d_{\mathcal{F}}(p, x) < \varepsilon\}$. Since $d_{\mathcal{F}}$ is continuous $B(p, \varepsilon)$ is open. Further, by Corollary 3.3, $B(p, \varepsilon)$ is a union of leaves of $\mathcal{F}$. Let $N_0$ and $N_1$ be two leaves of $\mathcal{F}$. To show that $M/\mathcal{F}$ is Hausdorff we must find disjoint open sets $O_0$ and $O_1$ containing $N_0$ and $N_1$ respectively, and which are the union of leaves.

Let $\delta = d_{\mathcal{F}}(N_0, N_1).$ Since $\mathcal{F}$ is hyperbolic $\delta > 0$.

Let $p_0 \in N_0, p_1 \in N_1$. Then the required sets are $O_0 = B(p_0, \delta/2)$ and $O_1 = B(p_1, \delta/2)$.

Combining Proposition 3.4 and Lemma 3.6 we have the following.

Theorem 3.7. Every hyperbolic foliation of a compact manifold is compact and Hausdorff.

Remark. By Epstein [4] and Theorem 3.7 a hyperbolic foliation of a compact manifold is locally given by a Seifert fibration. From this it follows that the leaf space has the structure of a hyperbolic $V$ manifold.

Definition. A Hermitian foliation is a holomorphic foliation with a transverse $J$-structure, compatible with the homomorphic structure.

Proposition 3.8. Let $\mathcal{F}$ be a complex, codimension-$q$, hyperbolic foliation of a compact manifold $M$. Then $\mathcal{F}$ is Hermitian.

Proof. Hamilton [5] has shown that every compact Hausdorff foliation is Riemannian. A Riemannian, holomorphic foliation is a $U(q)$-foliation.

Proposition 3.9. If $\mathcal{F}$ is a hyperbolic foliation of a compact manifold $M$, then all secondary invariants defined by the characteristic homomorphism
\[ \Delta_\ast : H^1 \left( W(\mathfrak{gl}(q, \mathbb{C}), U(q))_2 q \right) \to H_{DR}^1 (M, \mathbb{R}) \]

are trivial.

PROOF. This is immediate from Proposition 3.9 and Theorem 4.52 of Kamber-Tondeur [6].

BIBLIOGRAPHY


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