ON NONLINEAR VARIATIONAL INEQUALITIES

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Abstract. In this note we have given a direct proof of the result which states that if $K$ is a compact convex subset of a linear Hausdorff topological space $E$ over the reals and $T$ is a monotone and hemicontinuous (nonlinear) mapping of $K$ into $E^*$, then there is a $u_0 \in K$ such that $(T(u_0), v - u_0) > 0$ for all $v \in K$.

Introduction. Browder [1] has proved that if $K$ is a closed convex subset of a reflexive Banach space $E$ such that $0 \in K$ and $T$ is a monotone and hemicontinuous nonlinear mapping of $K$ into $E^*$ satisfying the coercivity condition, then there is a $u_0 \in K$ such that $(T(u_0), v - u_0) > 0$ for all $v \in K$. Hartman and Stampacchia [3] have independently proved a similar result and made applications to second order nonlinear elliptic equations. This result with $c(u) = 0$ (see Theorem 1.1 of [3]) is a special case of Browder's result [1]. With the closed convex subset $K$ of $E$ as assumed in [3], the coercivity condition on $T$ reduces the problem to proving the existence of $u_0$ satisfying the above inequality in a closed bounded convex subset of $K$ (see remark following Theorem 1.1 and Lemma 2.2 in [3]). Thus it is of interest to prove the above result in a weakly compact convex subset of an arbitrary Banach space. This would then contain Theorem 1.1 in [3] and the result of [1] as special cases. In fact the main object of this paper is to prove this result in a compact convex subset of a linear topological space over the reals without the coercivity condition on $T$. The techniques used in [1] and [3] are more or less the same, 'to prove the result in a finite dimensional case and then apply a limiting procedure'. We will give a direct proof of our result by applying a generalized version of a fixed point theorem of Browder [2].

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We first prove a slight generalization of a fixed point theorem of Browder [2, Theorem 1, p. 285] which will suit our purpose.

Theorem 1. Let $K$ be a nonempty compact convex subset of a Hausdorff linear topological space $E$. Let $T$ be a multivalued mapping of $K$ into $2^K$ such that

(i) for each $x \in K$, $T(x)$ is a nonempty convex subset of $K$;
(ii) for each $y \in K$, $T^{-1}(y) = \{x \in K : y \in T(x)\}$ contains an open subset
O_y of K (O_y may be empty);
(iii) \( \bigcup \{ O_y : y \in K \} = K \).

Then there exists a point \( x_0 \in K \) such that \( x_0 \in T(x_0) \).

**Proof.** Although the proof is similar to that in [2], we include it for the sake of completeness. Since K is compact, by (iii) there exists a finite family \( \{ y_1, y_2, \ldots, y_n \} \) such that \( K = \bigcup_{i=1}^n O_{y_i} \). Let \( \{ f_1, f_2, \ldots, f_n \} \) be a partition of unity corresponding to this finite covering, i.e. each \( f_i, i = 1, 2, \ldots, n, \) is a real valued continuous function defined on K such that \( f_i \) vanishes outside \( O_{y_i}, 0 < f_i(x) < 1, \) for all \( x \in K \) and \( \sum_{i=1}^n f_i(x) = 1 \) for each \( x \in K \).

We define a mapping \( p : K \to K \) by

\[
p(x) = \sum_{i=1}^n f_i(x) y_i, \quad x \in K.
\]

Obviously \( p \) maps K into K and is continuous. Also for each \( k \) with \( f_k(x) \neq 0, x \in O_{y_k} \subseteq T^{-1}(y_k), \) i.e. \( y_k \in T(x) \). As \( T(x) \) is convex, this implies that \( p(x) \in T(x) \) for each \( x \in K \).

Let \( S \) be the finite dimensional simplex spanned by \( y_1, y_2, \ldots, y_n \). Then clearly \( p \) maps \( S \) into \( S \). Also, since \( E \) is Hausdorff linear topological space, the topology on \( S \) induced by the topology in \( E \) is Euclidean. Hence by the Brouwer fixed point theorem, there is a point \( x_0 \in S \) such that \( x_0 = p(x_0) \in T(x_0) \).

Let \( K \) be a subset of a linear topological space \( E \) over the reals and \( T \) a single valued (nonlinear) mapping of \( K \) into \( E^* \). We say \( T \) is monotone provided \( (T(u) - T(v), u - v) > 0 \) for all \( u, v \in K \). Here \( (\cdot, \cdot) \) denotes the pairing between \( E^* \) and \( E \).

\( T : K \to E^* \) is said to be hemicontinuous if \( T \) is continuous from the line segments in \( K \) to the weak topology of \( E^* \).

A point \( u_0 \in K \) is said to satisfy the variational inequality if

\[
(T(u_0), v - u_0) > 0 \quad \text{for all } v \in K . . .
\]

\( u_0 \) is also called a solution of (1).

**Lemma.** If \( K \) is a convex subset of a linear Hausdorff topological space \( E \), and \( T \) is a single valued mapping of \( K \) into \( E^* \) such that \( T \) is monotone and hemicontinuous, then \( u_0 \) is a solution of (1) if and only if \( u_0 \) is a solution of

\[
(T(v), v - u_0) > 0 \quad \text{for all } v \in K . . .
\]

**Proof.** The proof of this lemma on a Banach space in [1, Lemma 1] or in [3, Lemma 2.3] also holds here. If \( u_0 \) satisfies (1), then an application of monotonicity shows that \( u_0 \) satisfies (2). Now suppose that \( u_0 \) satisfies (2). As in [1] and [3] we employ a device of Minty [4]. Let \( \epsilon \) be an arbitrary point of \( K \). Then since \( K \) is convex, \( \epsilon = (1 - \iota)u_0 + \iota v \in K \) for \( 0 < \iota < 1 \). By (2) we have
\[ 0 < (T(v), t(v - u_0)) = t(T(v), v - u_0). \]

Since \( t > 0 \), \( (T(v), t(v - u_0)) > 0 \).

Now letting \( t \to 0 \) and using hemicontinuity of \( T \), \( T(v) \to T(u_0) \) weakly in \( E^* \). Hence \( (T(u_0), v - u_0) \geq 0 \).

**Remark.** We note that in the proof of the first part the convexity of \( K \) is not needed. In fact, if \( T: K \to E^* \) is a monotone mapping of any set \( K \subseteq E \) into \( E^* \), then given \( u \in K \), the set \( \{v: (T(u), v - u) > 0\} \subseteq \{v: (T(v), v - u) > 0\} \).

This follows from the definition of monotonicity, i.e., \( (T(v), v - u) > (T(u), v - u) \).

**Theorem 2.** Let \( K \) be compact convex subset of a linear Hausdorff topological space \( E \). Let \( T \) be a (single valued) monotone (nonlinear) mapping of \( K \) into \( E^* \). Suppose further that

\[ (*) \text{ for each } v \in K \text{ there exists } u \in K \text{ such that } (T(u), u - v) < 0. \]

Then there is a solution \( u_0 \) of (1), i.e. there is \( u_0 \in K \) such that \( (T(u_0), v - u_0) > 0 \) for all \( v \in K \).

**Proof.** We assume that there is no solution of (1). Then for each \( u \in K \), the set \( \{v \in K: (T(u), v - u) < 0\} \) is nonempty. We define a multivalued mapping \( F: K \to 2^K \) by

\[ F(u) = \{v \in K: (T(u), v - u) < 0\}. \]

\( F(u) \) is nonempty and clearly convex for each \( u \in K \). We now consider

\[ F^{-1}(u) = \{v \in K: u \in F(v)\} = \{v \in K: (T(v), u - v) < 0\}. \]

For each \( u \in K, [F^{-1}(u)]^c \) is the complement of \( F^{-1}(u) \) in

\[ K = \{v: (T(v), u - v) > 0\} \subseteq \{v: (T(u), v - u) > 0\} \]

by monotonicity of \( T = B(u) \), say. Obviously \( B(u) \) is a closed and convex subset of \( K \). Thus the complement of \( B(u) = [B(u)]^c \) is open in \( K \). Since \( [F^{-1}(u)]^c \subseteq B(u) \), it follows that \( [B(u)]^c \subseteq F^{-1}(u) \). Thus for each \( u \in K \), \( F^{-1}(u) \) contains an open set \( [B(u)]^c \) of \( K \).

Now from the hypothesis that for each \( v \in K \), there exists \( u \in K \) such that \( (T(u), u - v) < 0 \), it follows that \( \bigcup \{[B(u)]^c, u \in K\} = K \). Thus \( F \) satisfies all the conditions of our Theorem 1. Hence there exists a point \( w \in K \) such that \( w \in F(w) \), i.e. \( 0 > (T(w), w - w) = 0 \), which is impossible.

**Corollary.** Let \( K \) be a compact convex subset of a linear Hausdorff topological space \( E \). Let \( T \) be a monotone and hemicontinuous (nonlinear) mapping of \( K \) into \( E^* \). Then there is a solution \( u_0 \) of (1), i.e., there is \( u_0 \in K \) such that \( (T(u_0), v - u_0) > 0 \) for all \( v \in K \).

**Proof.** If \( (*) \) of Theorem 2 holds, then we have a solution \( u_0 \) of (1) by Theorem 2. If \( (*) \) does not hold, then it means precisely that there is \( u_0 \in K \) such that \( (T(u), u - u_0) > 0 \) for all \( u \in K \). Since \( T \) is hemicontinuous, the lemma implies that \( (T(u_0), u - u_0) > 0 \) for all \( u \in K \), i.e. \( u_0 \) is a solution of the variational inequality.
Remark. It has already been pointed out in the introduction that our corollary contains the result of [1] and Theorem 1.1 of [3] as a special case. It is also worth noting that

(i) it follows from the proof of our theorem that we can replace the monotonicity condition by a weaker condition that for each \( u \in K \),
\[
\{ v: (T(v), u - v) \geq 0 \} \subseteq \{ v: (T(u), u - v) \geq 0 \};
\]

(ii) in case of a locally convex Hausdorff topological space \( E \), it does not matter whether we assume \( K \) to be compact or weakly compact. The corollary still remains true as \( T \) remains hemicontinuous in either case.

References


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