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A NOTE ON A PAPER OF J. D. STEIN, JR.

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ABSTRACT. Among other results, it is proved that if a sequence $\{\mu_n\}$ of regular measures on a Hausdorff space, with values in a normed group, is convergent to zero for all σ-compact sets or all open sets, then there exists a maximal open set U such that $\dot{\mu}_n(U) \to 0$, $\{\dot{\mu}_n\}$ being the associated submeasures.

In [5], J. D. Stein, Jr. considers some versions of Phillip's lemma and the Vitali-Hahn-Saks theorem for sequences of regular scalar-valued Borel measures on Hausdorff spaces. The measures he considers are bounded, regular, and finitely additive Borel measures which are easily seen to be countably additive (to prove this, first note that $|\mu|$, the variation of such a measure μ, is regular, bounded, and finitely additive [5, Lemma 1]) and so for a sequence $\{B_i\}$ of Borel sets, $B_i \downarrow \varnothing$, and $\varepsilon > 0$, $\in$ a sequence of compact sets $\{K_i\}$, $K_i \subset B_i$ and $|\mu|(B_i \setminus K_i) \leqslant \varepsilon/2^i$, $\forall i$ which means that

$$|\mu|(B_i \setminus C_i) \leqslant |\mu|\left(\bigcup_{n=1}^{i} (B_n \setminus K_n)\right) \leqslant \varepsilon, \quad \forall i,$$

where $C_i = \bigcap_{n=1}^{i} K_n$. Since $C_i \downarrow \varnothing$ we get $C_i = \varnothing$, $\forall i \geqslant$ some n_0 and so $|\mu|(B_i \downarrow 0)$, an observation which enables one to prove his results easily, in more general forms, and under weaker conditions.

Theory of submeasures developed in [1] will be used. Let G be an Abelian Hausdorff topological group, X a Hausdorff topological space, and μ a countably additive, regular, G-valued measure on X (by regularity we mean that for any Borel set B in X and a 0-nbd U in G, there exists a compact set $C \subset B$ such that $\mu(K) \in U$, $\forall K \subset B \setminus C$, K Borel). If G is normed [1, p. 270], we have an associated submeasure $\dot{\mu}$,

$$\dot{\mu}(B) = \sup\{|\mu(A)| : A \subset B, A \text{ Borel}\} = \sup\{|\mu(C)| : C \subset B, C \text{ compact}\},$$

which is finite [1, p. 279, Corollary 4.11], exhaustive, σ-subadditive, and order-continuous [1, II]. It is also regular in the sense that given $\varepsilon > 0$ and a Borel set B, $\exists$ a compact set $K \subset B$ such that $\dot{\mu}(B \setminus K) < \varepsilon$ (proof by contradiction, using exhaustivity). For a collection of measures or sub-

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measures, the meaning of uniform regularity is comprehensible. We denote by N the set of all natural numbers.

THEOREM 1. *Let* $\{\mu_n\}$ *be a sequence of countably additive, regular G-valued Borel measures on a Hausdorff topological space* X *(*G *being a normed Abelian topological group), convergent to zero for every open subset of* X *or every* σ*-compact subset of* X*. Then* $\bigcup\{V: V \in \mathfrak{U}\} \in \mathfrak{U}$ *where* $\mathfrak{U} = \{V: V$ *open in* X *and* $\dot{\mu}_n(V) \to 0\}$.

PROOF. By [4], $\{\mu_n\}$ is convergent to 0 for all Borel sets if it is convergent to zero for all open sets. By induction there exists an increasing sequence of compact sets $\{C_i\}$ such that $\dot{\mu}_n(X \setminus \bigcup_{i=1}^{\infty} C_i) = 0$, $\forall n$. Since for any closed subset K of X, $K \cap (\bigcup_{i=1}^{\infty} C_i)$ is σ-compact, $\{\mu_n\}$ is convergent to zero for every closed set, and therefore for every open set, if it is convergent to zero for σ-compact sets. Thus in all cases $\{\mu_n\}$ converges to 0 for all Borel sets and so $\{\mu_n\}$ are uniformly regular [3, Theorem 8, p. 354] and so $\{\dot{\mu}_n\}$ are uniformly exhaustive [1, II, 4.2, p. 277]. Putting $U = \bigcup\{V: V \in \mathfrak{U}\}$ and fixing $\varepsilon > 0$, we get, by uniform regularity of $\{\dot{\mu}_n\}$, a compact set $C \subset U$ such that $\dot{\mu}_n(U \setminus C) \leqslant \varepsilon/2$, $\forall n$. This means that there exists $\{U_i\}$, $1 \leqslant i \leqslant m$, in $\mathfrak{U}$, such $\bigcup_{i=1}^{m} U_i \supset C$, and so

$$\dot{\mu}_n(U) \leqslant \dot{\mu}_n(U \setminus C) + \dot{\mu}_n(C) \leqslant \varepsilon/2 + \sum_{i=1}^{m} \dot{\mu}_n(U_i),$$

which gives the required result.

REMARK. [5, Proposition 2.1] is a particular case of this theorem, since in case $\{\mu_n\}$ are scalar-valued $|\mu_n| \leqslant 4\dot{\mu}_n$. Also since $\{\dot{\mu}_n\}$ are uniformly regular, $\dot{\mu}_n(K) \to 0$ for every compact K implies $\dot{\mu}_n(X) \to 0$, [5, Proposition 2.3] is a particular case of this assertion. The uniform regularity of $\{\dot{\mu}_n\}$ further implies that given $\varepsilon > 0$ and an open set U, there exists a compact set $C \subset U$ such that $\dot{\mu}_n(U \setminus C) \leqslant \varepsilon$, $\forall n$; in particular this implies that for given $x \in X$, $\varepsilon > 0$, and an open nbd V of x, there exists a compact subset $C \subset V \setminus \{x\}$ such that $\dot{\mu}_n(V \setminus \{x\} \setminus C) \leqslant \varepsilon/2$, $\forall n$. Since $\dot{\mu}_n\{x\} = |\mu_n(\{x\})| \to 0$, there exists an n_0 such that $\dot{\mu}_n\{x\} \leqslant \varepsilon/2$, $\forall n \geqslant n_0$. This means

$$\dot{\mu}_n(V \setminus C) \leqslant \dot{\mu}_n(V \setminus C \setminus \{x\}) + \dot{\mu}_n\{x\} \leqslant \varepsilon, \quad \forall n \geqslant n_0.$$

[5, Proposition 3.1] is a very particular case of this, since $V \setminus C$ is an open nbd of the point x. Also, given $\varepsilon > 0$ and compact countable subset $C = \{x_1, x_2, \ldots\}$ of X, the above result gives, $\forall i$, an open nbd U_i of x_i and $n_i \in N$ such that $\dot{\mu}_n(U_i) < \varepsilon/2^i$, $\forall n \geqslant n_i$, $\forall i$. By compactness of C, there exist an open set $U \supset C$ and an $n_0 \in N$ such that $\dot{\mu}_n(U) \leqslant \varepsilon$, $\forall n \geqslant n_0$, which means that $\dot{\mu}_n(C) \to 0$. This gives

COROLLARY 2. *Suppose* $\{\mu_n\}$ *satisfy the conditions of Theorem* 1 *and are inner regular by compact countable subsets of* X*. Then* $\dot{\mu}_n(X) \to 0$.

PROOF. The only thing to verify is that X is inner regular by a countable compact set uniformly for $\{\dot{\mu}_n\}$. In case this is not true, there will exist, by taking subsequences, if necessary, a disjoint sequence $\{C_i\}$ of countable compact sets such that $|\mu_i(C_i)| \geqslant \varepsilon$, $\forall i$, for some $\varepsilon > 0$, which is impossible since $\{\mu_i\}$ are uniformly exhaustive. [5, Proposition 3.2], is a particular case of this.

THEOREM 3. *Let $\{\mu_n\}$ be a sequence of regular G-valued Borel measures on X, a Hausdorff topological space, convergent for all open sets or all σ-compact sets in X (G being any Abelian Hausdorff topological group), and λ a σ-subadditive submeasure on the Borel subsets of X such that each μ_n is λ-continuous (in the terminology of* [1]). *Then $\{\mu_n\}$ are equi-λ-continuous.*

PROOF. As in Theorem 1, the hypothesis implies that $\{\mu_n(B)\}$ is convergent for every Borel set B in X. The result follows now from [1, Theorem 3.2, p. 276].

REMARK. [5, Proposition 2.2] is a particular case of this and the maximal open set U of that proposition is X.

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