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NECESSARY AND SUFFICIENT TYPE THEOREM FOR ABSOLUTE NÖRLUND SUMMABILITY OF CONJUGATE SERIES

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ABSTRACT. Sufficient conditions for $|N, p_n|$ summability of the conjugate series of Fourier series have been obtained earlier. The author obtains here a set of necessary and sufficient conditions for the result. This improves earlier results and provides a unified version of them.

1. **Introduction and the main results.** In the present paper we shall prove a result which is directly connected with an earlier paper due to Dikshit and Kumar [3]. Unless stated otherwise we shall use throughout this paper the same definitions and notations as used in [3]. We shall also use the following additional notations: $\sum_{n=1}^{\infty} B_n(t)$ denotes the conjugate series of $\sum_{n=0}^{\infty} A_n(t)$; $\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$; and '$BV\ [a, b]$' represents the class of functions of bounded variation over $[a, b]$.

Throughout $\lambda(t)$ will denote a positive nondecreasing function; for $n = 1, 2, 3, \ldots, \lambda(n) = \lambda_n$. Further, we write

$$J_n(\lambda_n) = \sum_{k=n}^{\infty} (P_n \lambda_k / \lambda_n k P_k).$$

Combining a result due to Bosanquet and Hyslop [1] with a subsequent result due to Mohanty [5], we have the following.

THEOREM A. *Suppose that*

$$\int_0^{\pi} t^{-\alpha} |d\psi(t)| < \infty \quad \textit{for } 0 \leqslant \alpha < 1. \tag{1.1}$$

Then $\sum_{n=1}^{\infty} n^{\alpha} B_n(x)$ *is summable* $|C, \alpha + \delta|$ *for every* $\delta > 0$ *if*

$$\int_0^{\pi} t^{-\alpha-1} |\psi(t)|\, dt < \infty. \tag{1.2}$$

When $\alpha = 0$, *then* (1.2) *is also a necessary condition for the* $|C, \delta|$ *summability of* $\sum_{n=1}^{\infty} B_n(x)$ (*see* [1, *Theorem* 4]).

In view of the lemma [5, p. 66] the conditions

$$t^{-\alpha} \psi(t) \in BV[0, \pi] \tag{1.1'}$$

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and (1.2), for $0 < \alpha < 1$, are equivalent to the corresponding conditions used by Mohanty in [5, Theorem 2]. However, (1.1′) is equivalent to (1.1) whenever (1.2) holds. For, if (1.1) holds, then (cf. [1, p. 495])

$$\int_0^\pi |d\{t^{-\alpha}\psi(t)\}| \leqslant \int_0^\pi t^{-\alpha}|d\psi(t)| + \alpha\int_0^\pi |\psi(t)|t^{-\alpha-1}dt < \infty$$

and thus (1.1′) is valid. Conversely, assuming (1.1′), we have

$$\int_0^\pi t^{-\alpha}|d\psi(t)| \leqslant \int_0^\pi |d\{t^{-\alpha}\psi(t)\}| + \alpha\int_0^\pi |\psi(t)|t^{-\alpha-1}dt < \infty$$

and, therefore, (1.1) is valid.

The sufficiency part of Theorem A was generalised in the form of the following results (see [6], [2] respectively).

THEOREM B. *Suppose* (1.1) *and* (1.2) *hold with* $\alpha = 0$, *and let* $\{p_n\}$ *be a nonnegative nonincreasing sequence such that*

$$\sum_{n=1}^{\infty} |\Delta R_n| < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} |\Delta S_n| < \infty,$$

where $R_n = (n+1)p_n/P_n$ *and* $S_n = \Sigma_{k=1}^n P_k/kP_n$, *then* $\Sigma_{n=1}^\infty B_n(x)$ *is summable* $|N, p_n|$.

THEOREM C. *Let* $\{p_n\}$ *be such that* $p_n > 0$, $p_{n+1}/p_n \leqslant p_{n+2}/p_{n+1} \leqslant 1$ *for all* n. *Then, for* $0 < \alpha < 1$, $\Sigma_{n=1}^\infty n^\alpha B_n(x)$ *is summable* $|N, p_n|$, *if* (1.1), (1.2) *hold, and* $J_n(n^\alpha) \leqslant K$.

Considering a class of functions $\lambda(t)$, we shall show that a condition which in particular reduces to (1.2) of Theorem A, is a necessary condition for $|N, p_n|$ summability of $\Sigma_{n=1}^\infty \lambda_n B_n(x)$. As for the sufficiency part of the theorem, we obtain a sharper result which includes both Theorem B and Theorem C. We shall prove the following.

THEOREM. *Suppose that*

$$\int_0^\pi \lambda(C/t)|d\psi(t)| < \infty, \quad \textit{where } C \textit{ is a number} > \pi; \tag{1.3}$$

and let $\{p_n\}$ *be a nonnegative nonincreasing sequence such that*

$$J_n(\lambda_n) \leqslant K, \qquad n = 1, 2, \ldots, \tag{1.4}$$

then $\Sigma_{n=1}^\infty \lambda_n B_n(x)$ *is summable* $|N, p_n|$, *if and only if,*

$$\int_0^\pi \lambda(C/t)|\psi(t)|t^{-1}\,dt < \infty. \tag{1.5}$$

If we assume $\psi(0) = 0$, then $\psi(t) = \int_0^t d\psi(u)$, so that by a change in order of integration we get for $\alpha > 0$,

$$\int_0^\pi t^{-\alpha-1}|\psi(t)|\,dt \leqslant \int_0^\pi t^{-\alpha-1}\,dt \int_0^t |d\psi(u)| \leqslant \frac{1}{\alpha}\int_0^\pi u^{-\alpha}|d\psi(u)|.$$

Thus, if $\psi(0) = 0$ and $\alpha > 0$, then (1.1) implies (1.2). However, it may be observed that (1.3) does not imply (1.5) even if $\psi(0) = 0$. For if we take

$\lambda(u) = \log u$ and $\psi(t) = (\log(C/t))^{-2}$, then (1.3) is true but (1.5) is not satisfied.

2. **Preliminary results.** We need the following lemmas for proof of the theorem. We introduce the following notations for convenience: $\tau = [\pi/t]$; for each positive integer n, $P(n, k) = P_{n-k}/P_n - P_{n-k-1}/P_{n-1}$, where $0 \leqslant k \leqslant n$, $P(0, 0) = 1$.

LEMMA 1. *Let* $\{p_n\}$ *be a nonnegative nonincreasing sequence. Then* (i) *for all* $k \geqslant 0$, $\Sigma_{n=a}^{b} P(n, k) \leqslant 1$, *where* $1 \leqslant a \leqslant b \leqslant \infty$, *and* $P(n, k) \geqslant 0$;

(ii) $\Sigma_{k=0}^{n} P(n, k) \geqslant 1/2$ *for all* $n > 0$.

PROOF. The result (i) of the lemma is contained in ([3, Lemma 3]). Writing $P_n^1 = \Sigma_{k=0}^{n} P_n$, we have

$$\sum_{k=0}^{n} P(n, k) = \frac{P_n^1}{P_n} - \frac{P_{n-1}^1}{P_{n-1}} = 1 - \frac{P_{n-1}^1 p_n}{P_{n-1} P_n}.$$

In view of this equality, result (ii) is equivalent to

$$p_n P_{n-1}^1 \leqslant \tfrac{1}{2} P_{n-1} P_n, \tag{2.1}$$

which we now prove by induction. The result (2.1) is trivially true for $n = 1$. Assuming that the result (2.1) is true for n, we obtain the following by adding $p_n P_n$ to both the sides of (2.1).

$$p_n P_n^1 \leqslant \tfrac{1}{2} P_n (P_n + p_n). \tag{2.2}$$

Since $p_{n+1} - p_n \leqslant 0$ and $P_n^1 > \frac{1}{2} P_n$, we have

$$(p_{n+1} - p_n) P_n^1 \leqslant \tfrac{1}{2} P_n (p_{n+1} - p_n). \tag{2.3}$$

Adding (2.2) and (2.3), we see that (2.1) holds with n replaced by $(n + 1)$. Hence the result.

LEMMA 2. *If* $\{p_n\}$ *be nonnegative nonincreasing and* (1.4) *holds, then* (i) $\lambda_{2n} \leqslant K\lambda_n$, (ii) *for* $0 < k \leqslant n$, $k\lambda_n \leqslant Kn\lambda_k$ *and* (iii) $\Sigma_{k=1}^{n} (P_n - P_{n-k})/k \leqslant KP_n$.

PROOF. It is clear that (ii) includes (i) which is contained in [3, Lemma 4]. We, therefore, consider (ii). Since the nonnegative nonincreasing nature of $\{p_n\}$ implies that $\{P_n/n\}$ is nonincreasing, we have from (1.4)

$$\frac{K\lambda_k}{k} \geqslant \sum_{r=k}^{\infty} \frac{\lambda_r}{r^2} \geqslant \sum_{r=n}^{\infty} \frac{\lambda_r}{r^2} \geqslant \lambda_n \sum_{r=n}^{\infty} \frac{1}{r^2},$$

since $\{\lambda_n\}$ is nondecreasing. Thus, (ii) follows.

In order to prove (iii) we first observe that $S_n \leqslant K$ is equivalent to $J_n(1) \leqslant K$ (see [4, Lemma 4]) which is a direct consequence of (1.4). The result now follows when we observe that for nonnegative nonincreasing sequence $\{p_n\}$, $P_n - P_{n-k} \leqslant P_k$.

LEMMA 3. *If* $\{p_n\}$ *be nonnegative nonincreasing, and* (1.4) *holds, then uniformly for* $0 < t \leqslant \pi$

$$\sum_{n=2\tau+2}^{\infty} \left| \sum_{k=\tau+1}^{n} P(n,k)k^{-1}\lambda_k \exp(ikt) \right| \leqslant K\lambda(C/t),$$

where $\tau = [\pi/t]$.

The result of Lemma 3 follows from the proof of the result $\Sigma^* \leqslant K\lambda(C/t)$, as given in ([3, Proof of Theorem 1, (4.6)]).

LEMMA 4. *Suppose that* (1.3) *holds, and let for* $n > 1$, $E_n = \{t\colon |\psi(t)|$ *is continuous at* $t \in [\pi/n, \pi/(n-1))\}$. *Then there exist points* θ_n *and* θ_n' *in* E_n *such that, for all* $x \in E_n$,

$$|\psi(\theta_n)| - 1/n \leqslant |\psi(x)| \leqslant |\psi(\theta_n')| + 1/n. \tag{2.4}$$

Further, (i) *if* (1.4) *holds and* $\{p_n\}$ *is nonnegative nonincreasing, then* (1.5) *implies* $\Sigma_{n=2}^{\infty}\lambda_n|\psi(\theta_n)|/n < \infty$, *and* $\Sigma_{n=2}^{\infty}\lambda_n|\psi(\theta_n')|/n < \infty$ *implies* (1.5); *and* (ii) $\Sigma_{n=2}^{\infty}\lambda_n|\Delta\psi(\theta_n)| < \infty$.

PROOF. It is clear that (1.3) implies that $\psi(t) \in BV[0,\pi]$ so that $|\psi(t)| \in BV[0,\pi]$ and thus (2.4) follows directly. Further, since $|\psi(t)|$ possesses at most countable number of ordinary discontinuities, we have by the result (i) of Lemma 2

$$\frac{|\psi(\theta_n)|\lambda_n}{n} \leqslant K\int_{E_n} \left(|\psi(t)| + 1/n\right) \frac{\lambda(\pi/t)}{t}\, dt,$$

which gives that

$$\sum_{n=2}^{\infty} \frac{|\psi(\theta_n)|\lambda_n}{n} \leqslant K\int_0^{\pi} \frac{|\psi(t)|\lambda(\pi/t)}{t}\, dt + K\sum_{n=2}^{\infty} \frac{\lambda_n}{n^2}.$$

Applying the condition (1.4) along with the fact that $\{P_n/n\}$ is nonincreasing, we obtain the first result in (i). The second one can be obtained similarly.

In order to prove (ii), we apply result (i) of Lemma 2 so that

$$\lambda_n|\Delta\psi(\theta_n)| \leqslant K\lambda_{n-1}\left|\int_{\theta_{n+1}}^{\theta_n} d\psi(t)\right| \leqslant K\int_{\theta_{n+1}}^{\theta_n} \lambda(C/t)|d\psi(t)|,$$

since $\lambda(t)$ is nondecreasing. This proves (ii).

3. **Proof of the Theorem.** We have

$$B_k(x) = \frac{2}{\pi}\int_0^{\pi} \psi(t)\sin kt\, dt.$$

We first integrate by parts and then break the range of integration in $(0, \theta_n)$ and (θ_n, π) where θ's are the points introduced in Lemma 4. Thus, we have

$$\frac{k\pi}{2} B_k(x) = -\int_0^{\pi}(1 - \cos kt)d\psi(t)$$

$$= -\int_0^{\theta_n}(1 - \cos kt)d\psi(t) - \int_{\theta_n}^{\pi} d\psi(t) + \int_{\theta_n}^{\pi} \cos kt\, d\psi(t)$$

$$= -2\int_0^{\theta_n} \sin^2 \frac{kt}{2}\, d\psi(t) + \psi(\theta_n) + \int_{\theta_n}^{\pi} \cos kt\, d\psi(t).$$

Writing $V_n(a) = t_n - t_{n-1}$, where t_n's are (N, p_n) means of $a \equiv \Sigma_{n=1}^{\infty} \lambda_n B_n(x)$, we have

$$-\Sigma_1 + \Sigma_2 - \Sigma_3 \leqslant \frac{\pi}{2} \sum_{n=2}^{\infty} |V_n(a)| \leqslant \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{3.1}$$

where

$$\Sigma_1 = 2 \sum_{n=2}^{\infty} \left| \int_0^{\theta_n} \left\{ \sum_{k=1}^{n} \frac{P(n, k)\lambda_k \sin^2(kt/2)}{k} \right\} d\psi(t) \right|,$$

$$\Sigma_2 = \sum_{n=2}^{\infty} |\psi(\theta_n)| \sum_{k=1}^{n} \frac{P(n, k)\lambda_k}{k},$$

$$\Sigma_3 = \sum_{n=2}^{\infty} \left| \int_{\theta_n}^{\pi} \left\{ \sum_{k=1}^{n} \frac{P(n, k)\lambda_k \cos kt}{k} \right\} d\psi(t) \right|.$$

We first show that Σ_1 and Σ_3 are bounded. Since for the integral in Σ_1, $t \leqslant \theta_n < \pi/(n-1)$, we have

$$\Sigma_1 \leqslant 2\int_0^{\pi} |d\psi(t)| \sum_{n=2}^{\tau+1} \sum_{k=1}^{n} \frac{P(n, k)\sin^2(kt/2)\lambda_k}{k}.$$

Observing that $|\sin nt| \leqslant nt$, and applying results (i) of Lemma 1, Lemma 2, we have

$$\sum_{n=2}^{\tau+1} \sum_{k=1}^{n} \frac{P(n, k)\sin^2(kt/2)\lambda_k}{k}$$

$$\leqslant t^2 \sum_{k=1}^{\tau+1} k\lambda_k \sum_{n=k}^{\tau+1} P(n, k) \leqslant K\lambda_{\tau+1} \leqslant K\lambda(C/t).$$

Hence, by virtue of the hypothesis (1.3),

$$\Sigma_1 < \infty. \tag{3.2}$$

Now, we consider Σ_3. We write

$$g_n(t) = \sum_{k=1}^{n} P(n, k)\lambda_k k^{-1} \cos kt \quad \text{for } \theta_n \leqslant t \leqslant \pi,$$

otherwise zero. We have

$$\Sigma_3 \leqslant \sum_{n=2}^{\infty} \int_0^{\pi} |g_n(t)|\, |d\psi(t)|.$$

In order to prove that $\Sigma_3 < \infty$, we first demonstrate that

$$\Sigma' = \sum_{n=\tau}^{\infty} \left| \sum_{k=1}^{n} P(n, k)\lambda_k k^{-1}\exp(ikt) \right| \leqslant K\lambda(C/t),$$

uniformly in $0 < t \leqslant \pi$.

Breaking the range of summations suitably and observing that $|\exp(ikt)| = 1$, we have

$$\Sigma' \leqslant \sum_{n=\tau}^{\infty} \sum_{k=1}^{\tau} P(n, k)\lambda_k/k + \sum_{n=\tau+1}^{2\tau+1} \sum_{k=\tau+1}^{n} P(n, k)\lambda_k/k$$
$$+ \sum_{n=2\tau+2}^{\infty} \left| \sum_{k=\tau+1}^{n} P(n, k)\lambda_k k^{-1}\exp(ikt) \right|.$$

Using result (i) of Lemma 1, Lemma 2, together with the fact that $\{\lambda_n\}$ is nondecreasing, and Lemma 3, we have

$$\Sigma' \leqslant \sum_{k=1}^{\tau} \frac{\lambda_k}{k}\left(1 - \frac{P_{\tau-k-1}}{P_{\tau-1}}\right) + \sum_{k=\tau+1}^{2\tau+1} \frac{\lambda_k}{k} \sum_{n=k}^{2\tau+1} P(n, k) + K\lambda(C/t)$$
$$\leqslant K\lambda(C/t) + K\lambda(C/t) \leqslant K\lambda(C/t).$$

This result, by virtue of the hypothesis (1.3) and the fact that $g_n(t) = 0$ when $n \leqslant \tau$, yields that

$$\Sigma_3 < \infty. \tag{3.3}$$

Evidently,

$$|V_1(a)| = P(1, 1)|B_1(x)| < \infty. \tag{3.4}$$

In view of (3.1), (3.2), (3.3) and (3.4) it is enough for proving the theorem to show that $\Sigma_2 < \infty$, if and only if (1.5) holds. In order to establish the sufficiency part of the theorem, we first observe that $\psi(t) \to 0$ as $t \to 0$. For if $\psi(t) \to l \neq 0$, then in view of the fact that the positive nondecreasing property of $\lambda(t)$ implies the existence of a positive constant a such that $\lambda(C/t) \geqslant a$ in some right hand neighbourhood of the origin, it follows that (1.5) is contradicted. Now effecting suitable changes in order of summations and applying Abel's transformation, we get

$$\Sigma_2 = \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} |\psi(\theta_n)|P(n, k) + O(1)$$
$$= \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} \Delta|\psi(\theta_n)| \frac{P_{n-k}}{P_n} + O(1)$$
$$= \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} \Delta|\psi(\theta_n)| - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} \frac{P_n - P_{n-k}}{P_n} \Delta|\psi(\theta_n)| + O(1)$$
$$= \sum_{k=2}^{\infty} \frac{\lambda_k}{k} |\psi(\theta_k)| - \sum_{n=2}^{\infty} \Delta|\psi(\theta_n)| \sum_{k=2}^{n} \frac{(P_n - P_{n-k})\lambda_k}{kP_n} + O(1) < \infty,$$

by result (iii) of Lemma 2, Lemma 4 and the fact that $\{\lambda_n\}$ is nondecreasing.

This completes the proof of the sufficiency part of the theorem.

Next, by the results (ii) of Lemmas 1 and 2, we have

$$\Sigma_2 \geqslant K \sum_{n=2}^{\infty} \frac{|\psi(\theta_n)|\lambda_n}{n}$$

$$\geqslant -K \sum_{n=2}^{\infty} \left||\psi(\theta_n)| - |\psi(\theta_n')|\right| \frac{\lambda_n}{n} + K \sum_{n=1}^{\infty} \frac{|\psi(\theta_n')|\lambda_n}{n}.$$

Since by result (ii) of Lemma 2, $\lambda_n/n \leqslant K$, the first term on the right hand side is bounded by virtue of the fact that $|\psi(t)| \in BV[0, \pi]$, which is a consequence of (1.3). Now the necessity part of the theorem follows when we use result (i) of Lemma 4.

This completes the proof of the theorem.

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