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A BOUNDARY VALUE PROBLEM FOR $H^\infty(D)$

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ABSTRACT. Let $W = \bigcup_{n=1}^\infty W_n$ be an F_σ subset of the unit circle of measure 0 and let $\{q_n\}$, $n \geq 1$, be a decreasing sequence with $q_1 < 1$ and $\lim_n q_n = 0$. There exists an H in $H^\infty(D)$ of norm q_1 whose modulus has radial limit along every radius which has radial limit of modulus q_1 on W_1 and q_{n+1} on $W_{n+1} \setminus \bigcup_{k=1}^n W_k$. If W is simultaneously a G_δ set, H may be chosen to have no zeros on C. It follows that for W countable, say $W = \{e^{iw_n}\}$, $n \geq 1$, there is such an H of norm 1 for which $\lim_{r\to 1} H(re^{iw_n}) = 1/n$. The proof of the theorem depends on the existence of a special collection of closed sets $\{S_\lambda\}$, $\lambda \geq 1$, real, for which the function h, defined by $h(x) = a_n + [(\inf\{\lambda | x \in S\}) - n](a_{n+1} - a_n)$, $a_n = -\ln q_n$, is such that the function $H(w) = \exp(-1/2\pi)\int[(w + e^{iu})/(e^{iu} - w)]h(u)\,du$ has the required properties. Some of the techniques used are similar to those developed in an earlier paper [1].

NOTATION. Let I be a fixed interval on the real line. For each Lebesgue measurable subset E of I, $|E|$ denotes its Lebesgue measure. E' denotes the complement of E in I.

The letters k, m, n and s when they occur as symbols denote natural numbers. C denotes the unit circle and D the unit disc.

Purpose. The aim of this work is to prove the following theorem.

THEOREM. *Let $W = \bigcup_{n=1}^\infty W_n$ be an F_σ subset of the unit circle of measure* 0 *and let $\{q_n\}$, $n \geq 1$ be a strictly decreasing sequence with $q_1 < 1$ and $\lim_n q_n = 0$. There exists a function H in $H^\infty(D)$ of norm q_1 whose modulus has radial limit along every radius which has radial limit of modulus q_1 on W_1 and q_{n+1} on $W_{n+1} \setminus \bigcup_{k=1}^n W_k$. If W is simultaneously a G_δ set, then H may be chosen to have no zeros on C.*

An immediate corollary of the theorem is that for W countable, say $W = \{e^{iW_n}\}$, $n \geq 1$, there is such an H of norm 1 for which $\lim_{r\to 1} H(re^{iW_n}) = 1/n$.

Preliminary lemmas. Three lemmas precede the proof of the theorem. In addition, frequent reference is made to a lemma of Lusin-Menchoff. For convenience this lemma is stated first.

LUSIN-MENCHOFF LEMMA. *Let M_1 be an arbitrary nonempty measurable subset of* (0, 1). *Let M_2 be a closed subset of M_1 consisting only of points of*

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density of M_1. Then for every positive number p there exists a closed set M_p with $M_2 \subset M_p \subset M_1$ satisfying:

(1) *Every point of M_2 is a point of density of M_p and every point of M_p is a point of density of M_1.*

(2) $|M_p| > |M_2| + (1 - 2^{-2-p})(|M_1| - |M_2|)$.

(3) *If $x \in M_2$ and $|M_1 \cap (x, x+t)|/|t| > 1 - \varepsilon$ for $|t| < 1/m$, then $|M_p \cap (x, x+t)|/|t| > 1 - \varepsilon - 2^{-m-p+1}$, in particular, if $|M_1' \cap (0, 1)| = 0$, $|M_p \cap (x, x+t)|/|t| \geqslant 1 - 2^{-m-p+1}$ for $|t| < 1/m$.*

A proof of this lemma may be found in [2]. It is clear that the interval (0, 1) can be replaced by an arbitrary open interval (a, b). The interval may also be closed or half-open provided suitable one-sided restrictions are imposed in (1) and (3) at included endpoints. In the lemmas that follow such restrictions are assumed without explicit mention.

LEMMA 1. *Let $W = \bigcup_{n=1}^{\infty} W_n$, W_n closed, be an F_σ set of measure 0 contained in $[a, b]$. There exists an increasing sequence of closed sets $\{S_n\}_{n \geqslant 1}$ for which*

(a) $|S_n| > L(1 - 1/2^n)$ *for all n, where $L = b - a$.*

(b) $W_1 \subset S_1$ *and* $W_n \setminus \bigcup_{k=1}^{n-1} W_k C S_n \setminus S_{n-1}$ *for $n > 1$.*

(c) *For each $w \in W_n \setminus \bigcup_{k=1}^{n-1} W_k$, $n > 1$, there exists a $T_w > 0$ such that $|S_n \cap (w, w+t)| = 0$ for all t with $|t| < T_w$.*

(d) *For each n there exists a constant $P_n > n$ such that $|S_{n+1} \cap (x, x+t)|/|t| > 1 - 2^{-m-P_n+1}$ whenever $x \in S_n$ and $|t| < 1/m$.*

If W is simultaneously a G_δ set, then the S_n may be chosen so that, in addition, $\bigcup_n Sn = [a, b]$.

PROOF. Set $L = b - a$. If W is also a G_δ set, then W' is an F_σ set and may be written as $W' = \bigcup_{n=1}^{\infty} F_p$, F_p closed.

Let P_1 be an arbitrary closed subset of W' with $|P_1| > L(1 - \frac{1}{2})$ and set $S_1 = P_1 \cup W_1$. If W is a G_δ set, set $S_1 = P_1 \cup W_1 \cup F_1$. Let p_1 be a number greater than 1 for which $|S_1| + (1 - 2^{-2-p_1})(L - |S_1|) > L(1 - 1/2^2)$.

Since S_1 is a subset of $W' \cup W_1$, there exists a closed set P_2 with $S_1 \subset P_2 \subset W' \cup W_1$ satisfying properties (1)–(3) of the Lusin-Menchoff Lemma with $M_1 = W' \cup W_1$, $M_2 = S_1$ and $p = p_1$.

Set $S_2 = P_2 \cup W_2 \setminus W_1$. Since W_1 is a subset of S_1 and hence, of P_2, $S_2 = P_2 \cup W_2$, which is closed. If W is also a G_δ set, set $S_2 = P_2 \cup F_2 \cup W_2 \setminus W_1$.

Continue defining sets S_n, $n > 2$, inductively as follows. Let p_{n-1} be a number greater than $n - 1$ for which $|S_{n-1}| + (1 - 2^{-2-p_{n-1}})(L - |S_{n-1}|) > L(1 - 1/2^n)$.

By the Lusin-Menchoff Lemma, there is a closed set P_n with $S_{n-1} \subset P_n \subset W' \cup \bigcup_{k=1}^{n-1} W_k$ satisfying (1)–(3) with $M_1 = W' \cup \bigcup_{k=1}^{n-1} W_k$, $M_2 = S_{n-1}$ and $p = p_{n-1}$. Set $S_n = P_n \cup W_n \setminus \bigcup_{k=1}^{n-1} W_k$. If W is a G_δ set, set $S_n = P_n \cup F_n \cup W_n \setminus \bigcup_{k=1}^{n-1} W_k$. The sequence $\{S_n\}$, $n \geqslant 1$, has all the required properties.

Property (b) follows immediately from the construction of the S_n. Property (a) holds since $|S_n| > |S_{n-1}| + (1 - 2^{-2-p_{n-1}})(L - |S_{n-1}|)$, $n > 1$, by (2) of the Lusin-Menchoff Lemma, and the right side of the inequality is greater than $L(1 - 1/2^n)$ by choice of p_n. If $w \in W_n \setminus \bigcup_{k=1}^{n-1} W_k$, $n > 1$, then w is not in the closed set P_n (or in $P_n \cup F_n$) since $P_n \subset W' \cup \bigcup_{k=1}^{n-1} W_k$. Let T_w be the distance from w to P_n (or to $P_n \cup F_n$). Then for $|t| < T_w$, $|S_n \cap (w, w + t)| = |W_n \setminus \bigcup_{k=1}^{n-1} W_k \cap (w, w + t)| = 0$. Thus (c) holds. Finally, if $W' = \bigcup_{n=1}^{\infty} F_n$, F_n closed, S_n is chosen so that $F_n \subset S_n$. Consequently, $\bigcup_n S_n$ contains both W and W' and must equal $[a, b]$. Q.E.D.

It should be noted that an additional restriction may be imposed on the p_k, which will be used later. If $\{a_k\}$, $k \geqslant 1$, is a strictly increasing sequence of nonnegative numbers diverging to ∞, the p_k may be chosen so that $\sum_{k \geqslant 3} a_k 2^{-p_{k-2}} < \infty$. Choose, for example, each p_{k-2} large enough so that $a_k 2^{-p_{k-2}} < 1/2^k$.

An increasing sequence of closed sets S_n having properties (a)–(d) of Lemma 1 will be referred to as a modified Zahorski sequence for W on $[a, b]$.

For each N and $m/2^n$ with $0 < m < 2^n$, it is possible, by repeated use of the Lusin-Menchoff Lemma to insert a closed set $S_{N+m/2^n}$ between S_N and $S_{N+1} \setminus (W_{N+1} \setminus \bigcup_{k=1}^{n} W_k)$ in such a way that a set with smaller subscript is contained in and consists only of points of density of a set with larger subscript. These sets may be chosen so that, in addition, $|S_{N+1/2^n} \cap (x, x + t)|/|t| > L(1 - 2^{-m+1})$ for $x \in S_N$ whenever $|t| < 1/m$. Details for a similar situation may be found in [1, p. 166]. The essential difference in the present work is that the sets $S_{N+m/2^n}$, $m < 2^n$, are all contained in $S_{N+1} \setminus (W_{N+1} \setminus \bigcup_{k=1}^{n} W_k)$. Consequently, $W_{n+1} \setminus \bigcup_{n=1}^{n} W_k \subset S_{N+1} \setminus S_{N+m/2^n}$ for each N and all $m < 2^n$.

As in [1] for each $\lambda \geqslant 1$, let $S_\lambda = \bigcap_{N+m/2^n \geqslant \lambda} S_{N+m/2^n}$.

This collection of sets $\{S_\lambda\}_{\lambda \geqslant 1}$ will be referred to as a modified Zahorski collection for W on $[a, b]$.

LEMMA 2. *Let $W = \bigcup_{n=1}^{\infty} W_n$ be an F_σ set of measure 0 contained in the interval $[a, b]$ and let $\{a_n\}$, $n \geqslant 1$, be a strictly increasing sequence of nonnegative numbers diverging to ∞. Let $\{S_\lambda\}_{\lambda \geqslant 1}$ be a modified Zahorski collection for W on $[a, b]$ where the p_k are chosen so that $\sum_{k=3}^{\infty} a_k 2^{-p_{k-2}} < \infty$.*

For each $x \geqslant 1$, let $f(x) = a_n + (x - n)(a_{n+1} - a_n)$, where n is the unique positive integer for which $n \leqslant x < n + 1$. For each $x \in [a, b]$ let

$$h(x) = \begin{cases} \inf\limits_{\lambda > 1} \{\lambda | x \in S_\lambda\}, & x \in \bigcup\limits_{\lambda} S_\lambda, \\ \infty, & x \notin \bigcup\limits_{\lambda} S_\lambda. \end{cases}$$

Define the function g on $[a, b]$ by

$$g(x) = \begin{cases} f(h(x)) & \text{if } x \in \bigcup_\lambda S_\lambda, \\ \infty & \text{if } x \notin \bigcup_\lambda S_\lambda. \end{cases}$$

(If W is also a G_δ set, $\bigcup_\lambda S_\lambda$ may be the entire interval $[a, b]$, in which case the second part of the definition is superfluous.) The function g has these properties.

(a) g is bounded below by a_1 and is identically a_1 on S_1.

(b) $g(w) = a_{N+1}$ for each $w \in W_{N+1} \setminus \bigcup_{k=1}^N W_k$.

(c) g is upper-semicontinuous at each $x \in \bigcup_\lambda S_\lambda$ and $\lim_{u\to x} g(u) = \infty$ for each $x \notin \bigcup_\lambda S_\lambda$.

(d)

$$\lim_{t\to 0} \frac{1}{t} \int_x^{x+t} g(u)\, du = \begin{cases} g(x) & \text{for } x \in \bigcup_\lambda S_\lambda, \\ \infty & \text{for } x \notin \bigcup_\lambda S_\lambda. \end{cases}$$

PROOF. (a) and (b) follow immediately since $W_1 \subset S_1$ and $W_{N+1} \setminus \bigcup_{k=1}^N W_k \subset S_{N+1} \setminus S_{N+m/2^n}$ for each N and $m < 2^n$.

The function g is upper-semicontinuous at each x for which $g(x) = a_1$ since g is bounded below by a_1. Suppose that $g(x) > a_1$ and finite. Then $h(x) > 1$. Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be such that $h(x) - \delta \geqslant 1$ and $f(h(x) - \delta) > f(h(x)) - \varepsilon$. Since $h(x) = \inf_{\lambda \geqslant 1}\{\lambda | x \in S_\lambda\}$, $x \notin S_{h(x)-\delta}$. Let $t > 0$ be such that $(x - t, x + t) \cap S_{h(x)-\delta}$ is empty. Such a t exists since $S_{h(x)-\delta}$ is closed. For $u \in (x - t, x + t)$, $h(u) \geqslant h(x) - \delta$ and $g(u) \geqslant f(h(x) - \delta) \geqslant f(h(x)) - \varepsilon = g(x) - \varepsilon$. Thus g is upper-semicontinuous at x. If $x \notin \bigcup_\lambda S_\lambda$, then $h(u) > n$ and $g(u) > a_n$ for $u \in (x - t, x + t)$, $t = \operatorname{dis}(x_1, S_n)$. Thus $\lim_{u\to x} g(u) = \infty$. Therefore (c) holds.

It remains to verify (d)–a crucial property of g for future applications. (d) follows immediately from (c) if $x \notin \bigcup_{\lambda \geqslant 1} S_\lambda$. Let $x \in \bigcup_{\lambda \geqslant 1} S_\lambda$ and let n be the smallest positive integer for which $x \in S_n$. For every $0 < \varepsilon < (n + 1) - h(x)$ and any t,

$$\begin{aligned} \left| \frac{1}{t} \int_x^{x+t} g(u) - g(x)\, du \right| \leqslant & \left| \frac{1}{t} \int_{S_{h(x)+\varepsilon} \cap (x, x+t)} g(u) - g(x)\, du \right| \\ & + \left| \frac{1}{t} \int_{S_{n+1} \setminus S_{h(x)+\varepsilon} \cap (x, x+t)} g(u) - g(x)\, du \right| \\ & + \sum_{k=n+2}^{\infty} \frac{1}{|t|} \int_{S_k \setminus S_{k-1} \cap (x, x+t)} g(u) - g(x)\, du. \end{aligned}$$

Since $h(u) \leqslant k$ for $u \in S_k \setminus S_{k-1}$, $g(u) \leqslant a_k$ for $u \in S_k \setminus S_{k-1}$. Consequently,

$$\sum_{k=n+2}^{\infty} \frac{1}{|t|} \int_{S_k \setminus S_{k-1} \cap (x, x+t)} g(u) - g(x)\, du$$
$$\leqslant \sum_{k=n+2}^{\infty} (a_k - g(x)) \frac{|S_k \setminus S_{k-1} \cap (x, x+t)|}{|t|}.$$

It follows from property (d) of Lemma 1 and from the fact that x is in S_{k-2} for $k \geqslant n+2$, that the right side of this inequality is dominated by

$$\sum_{k=n+2}^{\infty} (a_k - g(x)) \frac{|t|_2^{-P_{k-2}-m+1}}{|t|} = 2^{-m+1} \sum_{k=n+2}^{\infty} (a_k - g(x)) 2^{-P_{k-2}}$$

when $|t| < 1/m$. This sum approaches 0 as $m \to \infty$ since p_{k-2} and $\sum_{k=n+2}^{\infty} a_k 2^{-p_{k-2}} < \infty$.

$$\left| \frac{1}{t} \int_{S_{n+1} \setminus S_{h(x)+\varepsilon} \cap (x, x+t)} g(u) - g(x)\, du \right|$$

also approaches 0 as $t \to 0$ since this expression is less than or equal to

$$(a_{n+1} - g(x)) |S_{n+1} \setminus S_{h(x)+\varepsilon} \cap (x, x+t)/|t||$$

and x is a point of density of both S_{n+1} and $S_{h(x)+\varepsilon}$. ($x \in S_{h(x)+\varepsilon/2}$ implies that x is a point of density of $S_{h(x)+\varepsilon}$.)

Finally, consider

$$\left| \frac{1}{t} \int_{S_{h(x)+\varepsilon} \cap (x, x+t)} g(u) - g(x)\, du \right|.$$

Let $|t|$ be small enough so that $g(u) \geqslant g(x) - \varepsilon$ for $u \in (x-t, x+t)$. Then,

$$(g(x) - \varepsilon) \left| \frac{S_{h(x)+\varepsilon} \cap (x, x+t)}{|t|} \right| \leqslant \frac{1}{|t|} \int_{S_{h(x)+\varepsilon} \cap (x, x+t)} g(u)\, du$$
$$\leqslant f(h(x) + \varepsilon) |S_{h(x)+\varepsilon} \cap (x, x+t)|/|t|.$$

Consequently,

$$\lim_{t \to 0} \left| \frac{1}{t} \int_{S_{h(x)+\varepsilon} \cap (x, x+t)} g(u) - g(x)\, du \right| \leqslant \max\{\varepsilon, f(h(x) + \varepsilon) - g(x)\}.$$

Summarizing, we have

$$\lim_{t \to 0} \left| \frac{1}{t} \int_x^{x+t} g(u) - g(x)\, du \right| \leqslant \max\{\varepsilon, f(h(x) + \varepsilon) - g(x)\}.$$

Since ε was arbitrary and f is continuous at $h(x)$, this limit must be 0.

This completes the proof of Lemma 2. Q.E.D.

LEMMA 3. *Let $W = \bigcup_{n=1}^{\infty} W_n$ be an F_σ set of measure 0 contained in the open interval (a, b). Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of nonnegative numbers diverging to ∞. Let $\{S_\lambda\}_{\lambda \geqslant 1}$ be a modified Zahorski collection for W on $[a, b]$ where the P_k are chosen so that $\sum_{k=3}^{\infty} a_k 2^{-p_{k-2}} < \infty$. If f, h and g are defined as in Lemma 2,*

$$\left|\int_0^R (1/t^2)\{J((w, w+t)) - J((w-t, w))\}\, dt\right|$$

is finite for every $w \in W$ *and positive number* R *less than* $\min(b - w, w - a)$. *Here* $J((w, w+t)) = \int_{(w,w+t)} g(u)\, du$ *and* $J((w-t, w)) = \int_{(w-t,w)} g(u)\, du$.

PROOF. The crucial observation here is that for each $w \in W_n \setminus \bigcup_{k=1}^{n-1} W_k$, $n > 1$, there exists a $T_w > 0$ such that $|S_n \cap (w, w+t)| = 0$ for all t with $|t| < T_w$. This is just (c) of Lemma 1. Consequently, $n \leqslant h(u) \leqslant n + 1/2^k$ for almost all u in the set $S_{n+1/2^k}(w-t, w+t)$ and $a_n \leqslant g(u) \leqslant a_n + (a_{n+1} - a_n)/2^k$ for almost all u in this set. Apart from the critical role played by this observation, the structure of the proof is very similar to that of Lemma 3 in [1]. Consequently, some details are omitted.

Fix R and w. For each measurable subset M of $(0, R)$ define $J(M) = \int_M g(u)\, du$, $M(t, +) = M \cap (w, w+t)$, $M(t, -) = M \cap (w-t, w)$ and $M_k = S_k \setminus S_{k-1}$.

If $w \in W_1$, then $w \in S_1$ and the result follows from Lemma 3 in [1]. Suppose that $w \notin W_1$ and let n be such that $w \in W_n \setminus \bigcup_{k=1}^{n-1} W_k$. Set

$$* = \int_0^R \left(\frac{1}{t^2}\right)\{J((w, w+t)) - J((w-t, w))\}\, dt,$$

$$A = \int_0^R \left(\frac{1}{t^2}\right)\left\{\sum_{k=n+2}^{\infty} J(M_k(t, +)) - J(M_k(t, -))\right\} dt,$$

$$B = \int_0^R \left(\frac{1}{t^2}\right)\{J(S_{n+1}(t, +)) - J(S_{n+1}(t, -))\}\, dt.$$

Note that $* = A + B$. Consider A first. For $u \in S_k$, $h(u) \leqslant k$ and $g(u) \leqslant a_k$. Thus

$$|A| \leqslant \int_0^R \left(\frac{1}{t^2}\right)\left\{\sum_{k=n+2}^{\infty} a_k(|M_k(t, +)|) + |M_k(t, -)|\right\} dt.$$

Using (d) of Lemma 1, it can be shown that

$$|A| \leqslant 4 \sum_{k=n+2}^{\infty} a_k 2^{-p_{k-2}} \int_0^R \left(\frac{-1/2^{\sqrt{2}\, t}}{t}\right) dt,$$

which converges since $\sum_{k \geqslant 3} a_k 2^{-p_{k-2}} < \infty$.

Now consider B. To show that $|B|$ is finite, it suffices to show that the integral converges with R replaced by T_w, where $T_w > 0$ is such that $|S_n \cap (w, w+t)| = 0$ for $|t| < T_w$. Fix $0 < t < T_w$ and let k be the positive integer for which $1/2^{k+1} < t^2 \leqslant 1/2^k$. Set $M(t) = (1/t^2)\{J(S_{n+1}(t, +)) - J(S_{n+1}(t, -))\}$ and set $B_S = S_{n+s/2^k} \setminus S_{n+(s-1)/2^k}$, $s > 1$. Then

$$M(t) = \left(\frac{1}{t^2}\right)\left\{J(S_{n+1/2^k}(t,+)) - J(S_{n+1/2^k}(t,-)) + \sum_{s=2}^{2^k} J(B_S(t,+)) - J(B_S(t,-))\right\}.$$

From the observation at the beginning of the proof, it follows that

$$J(S_{n+1/2^k}(t,+)) - J(S_{n+1/2^k}(t,-)) \leq a_n + 2^{-k}(a_{n+1} - a_n)|S_{n+1/2^k}(t,+)| - a_n|S_{n+1/2^k}(t,-)|.$$

The right side of this inequality is in turn dominated by

$$2^{-k}(a_{n+1} - a_n)|S_{n+1/2^k}(t,-)| + (a_n + 2^{-k}(a_{n+1} - a_n))(|S_{n+1/2^k}(t,+)| - |S_{n+1/2^k}(t,-)|)$$

The remaining details are essentially the same as those in the proof of Lemma 3 in [**1**, p. 172]. Q.E.D.

PROOF OF THEOREM. If $e^{i\pi} \notin W$, replace W_1 by $W_1 \cup \{e^{i\pi}\}$. Let $W_n^* = \{x \in [-\pi, \pi] | e^{ix} \in W_n\}$ and set $W^* = \bigcup_{n=1}^{\infty} W_n^*$. Set $a_n = -\ln q_n$. Let $\{S_\lambda\}_{\lambda \geq 1}$ be a modified Zahorski collection for W^* on $[-\pi, \pi]$ with the p_k so chosen that $\sum_{k=3}^{\infty} a_k 2^{-p_{k-2}} < \infty$.

If W is also a G_δ set, let $\{S_\lambda\}_{\lambda \geq 1}$ be such that $\bigcup_\lambda S_\lambda = [-\pi, \pi]$. Finally, let g be the function defined in Lemma 1 and extended periodically to a function on C. Define

$$H(w) = \exp\left(\frac{-1}{2\pi}\right)\int_{-\pi}^{\pi} \frac{e^{iu} + w}{e^{iu} - w}\, g(u)\, du, \qquad w \in D.$$

Since

$$\lim_{t \to 0} \frac{1}{t}\int_x^{x+t} g(u)\, du = \begin{cases} g(x), & x \in \bigcup_{\lambda \geq 1} S_\lambda, \\ \infty, & x \notin \bigcup_{\lambda \geq 1} S_\lambda, \end{cases}$$

$$\lim_{r \to 1} |H(re^{ix})| = \exp(-g(x)) \quad \text{for all } x \in [-\pi, \pi].$$

Since $g(x) = -\ln q_1$ for $x \in W_1^*$ and $g(x) = -\ln q_{n+1}$ for $x \in W_{n+1}^* \setminus \bigcup_{k=1}^{n} W_k^*$, this limit is q_1 on W_1 and q_{n+1} on $W_{n+1} \setminus \bigcup_{n=1}^{k} W_k$. If $\bigcup_\lambda S_\lambda = [-\pi, \pi]$, then the limit is clearly never 0.

To show that the radial limit of H itself exists on W_1 it suffices to show that $\int_0^\pi J((w, w+t)) - J((w-t, w))/t^2\, dt$ is finite for all $w \in W$ [**1**, p. 173]. The finiteness of this integral follows from Lemma 3. Q.E.D.

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