ON JAMES’ QUASI-REFLEXIVE BANACH SPACE

P. G. CASAZZA, BOR-LUH LIN AND R. H. LOHMAN

Abstract. In the James’ space $J$, there exist complemented reflexive subspaces which are not uniformly convexifiable and there are uncountably many mutually nonequivalent unconditional basic sequences in $J$ each of which spans a complemented subspace of $J$. If $(y_n)$ is a block basic sequence with constant coefficients of the unit vector basis of $J$, then its closed linear span $[y_n]$ is complemented in $J$ and the space $[y_n]$ is either isomorphic to $J$ or to $(\ell^1, \ell_1)$ for some $(\ell_n)$ where $J_n = [e_1, e_2, \ldots, e_n]$.

Introduction. For a sequence $x = (\alpha_1, \alpha_2, \ldots)$ of real numbers let

$$
\|x\| = \sup \left\{ \sum_{i=1}^{k} \left( \alpha_{p_{i+1}} - \alpha_{p_i} \right)^2 \right\}^{1/2}
$$

where the sup is over all positive integers $k$ and all increasing sequences $(p_i)$ of positive integers. Let $J$ be the Banach space of all $x$ such that $\|x\|$ is finite and $\lim_{n} \alpha_n = 0$. This remarkable Banach space, with an equivalent norm, was constructed by R. C. James [8], [9] to give an example of a separable Banach space whose canonical embedding in its second conjugate space $J^{**}$ is of codimension one. The study of $J$, on the one hand, leads to the study of shrinking (resp. boundedly complete) basic sequences in Banach spaces and to the study of Banach spaces which contain subspaces isomorphic to $c_0$ (resp. $\ell_1$). On the other hand, it leads to the study of quasi-reflexive spaces and is a major steppingstone to the study of nonreflexive Banach spaces (e.g., [3]). Recently, James [10], [11] solved two open problems in Banach space structure theory by constructing two ingenious examples of nonreflexive Banach spaces, both of which are closely related to $J$. Thus a better understanding of the structure of $J$ should aid in understanding the pathology that can occur in a nonreflexive Banach space.

The major known properties of $J$ are proved in [4], [5], [7], [8], [9]. In [2], a general construction of spaces of the same type as $J$ is introduced. Throughout this paper, we shall let $N$ be the set of all positive integers and let $\{e_n\}$ be the unit vector basis of $J$. For a sequence of elements $(\gamma_i)$ in $J$, we shall let $[\gamma_i]$ denote the closed linear subspace spanned by $(\gamma_i)$ in $J$. For the terminology on bases in Banach spaces, we refer the reader to [13].
In this paper, we show that there exist complemented reflexive subspaces in $J$ which are not uniformly convexifiable, hence there are complemented reflexive subspaces in $J$ other than $l_2$. It is well known that $J$ does not have an unconditional basis. However, we show that there are uncountably many mutually nonequivalent unconditional basic sequences in $J$ and every subsymmetric basic sequence in $J$ is equivalent to the unit vector basis of $l_2$.

Finally, although the basis $\{e_n\}$ is conditional, its block basic sequences with constant coefficients behave like those of a symmetric basis; namely, if $\{y_n\}$ is a block basic sequence with constant coefficients of $\{e_n\}$, then $[y_n]$ is complemented in $J$. Furthermore, $[y_n]$ is either isomorphic to $J$ or to $(\Sigma_{n=1}^\infty J_n)_2$ for some sequence $\{t_n\}$ in $N$, where $J_n = [e_1, \ldots, e_n]$, $n = 1, 2, \ldots$.

1. Reflexive subspaces. In this section, we show that there exist complemented reflexive subspaces of $J$ which are not uniformly convexifiable.

Definition. A sequence $\{n_i\}$ in $N$ is called proper if the complement $N \setminus \{n_i\}$ of $\{n_i\}$ in $N$ is infinite.

We prove that for any proper sequence $\{n_i\}$, the space $[e_{n_i}]$ is reflexive and is either isomorphic to $l_2$ or is not uniformly convexifiable.

Theorem 1. For any sequence $\{t_k\}$ in $N$, the natural basis of $(\Sigma_{k=1}^\infty J_{t_k})_2$ is equivalent to $\{e_{n_i}\}$ where $\{n_i\}$ is the sequence complementary to $\{t_{k+i}\}$ in $N$.

We first prove a technical lemma concerning the norm of $J$.

Lemma 2. Let $m_1 < n_1 < m_2 < n_2 < \cdots$ be a monotone increasing sequence of positive integers such that $m_k + 1 - n_k > 2$ for all $k$. For any real numbers $\alpha_i$,

$$\sum_{j=1}^k \left[ \sum_{i=m_j}^{n_j} \alpha_i e_i \right]^2 < \sum_{j=1}^k \left[ \sum_{i=m_j}^{n_j} \alpha_i e_i \right]^2 < 2 \sum_{j=1}^k \left[ \sum_{i=m_j}^{n_j} \alpha_i e_i \right]^2. \tag{2}$$

Proof. The first inequality in (2) is immediate since $\|\Sigma_{i=m_j}^{n_j} \alpha_i e_i\|$ can be evaluated for each $j$ by using only indices in the interval $[m_j - 1, n_j + 1]$. The second inequality follows from the fact that

$$(a - b)^2 < 2[(a - 0)^2 + (0 - b)^2],$$

which can be used to change any $(1)$-estimate of the middle member of (2) into a sum not larger than the last number.

Proof of Theorem 1. For any sequence $\{t_k\}$ in $N$, let $m_0 = 0$ and $m_k = \Sigma_{i=1}^{t_k} t_i + k$, $k \in N$. Let $\{n_i\}$ be the sequence complementary to $\{m_k\}$ in $N$. Note that $\{n_i\}$ is a proper sequence. For any element of the form

$$x = \left( \sum_{i=1}^{t_1} \alpha_{i(1)} e_i, \ldots, \sum_{i=1}^{t_1} \alpha_{i(t_1)} e_i, 0, 0, \ldots \right),$$
in \((\sum_{k=1}^{\infty} J_{k})_{i_2}\), let \(y = \sum_{k=1}^{n} \sum_{i=1}^{\infty} a_{i}^{(k)} e_{m_{i-1}+i}\). By Lemma 2, \(\|x\| < \|y\| < \sqrt{2} \|x\|\). It follows easily that \(\{e_{n}\}\) is equivalent to the natural basis of \((\sum_{k=1}^{\infty} J_{k})_{i_2}\).

Corollary 3. If \(\{n_{i}\}\) is a proper sequence then \([e_{n}]\) is a reflexive subspace of \(J\).

**Proof.** Let \(m_{0} = 0\) and let \(\{m_{k}\}\) be the sequence complementary to \(\{n_{i}\}\) in \(N\). If we define \(t'_{k} = m_{k} - m_{k-1} - 1\), then, after discarding those \(t'_{k}\) that are zero, the same argument as in the proof of Theorem 1 shows that there exists a sequence \(\{t_{k}\}\) in \(N\) such that \(\{e_{n}\}\) is equivalent to the natural basis of \((\sum_{k=1}^{\infty} J_{k})_{i_2}\). As the latter space is reflexive, the proof is complete.

It is well known that every basis in a uniformly convex space is \(p\)-Besselian for some \(1 < p < \infty\). On the other hand, if \(\lim \sup_{k} t_{k} = \infty\), the natural basis of \((\sum_{k=1}^{\infty} J_{k})_{i_2}\) is clearly not \(p\)-Besselian. Hence, by Corollary 3, the following is immediate.

**Corollary 4.** For any proper sequence \(\{n_{i}\}\), let \(\{m_{k}\}\) be the sequence complementary to \(\{n_{i}\}\) in \(N\). If

\[
\lim \sup_{k} (m_{k+1} - m_{k}) = \infty,
\]

then \([e_{n}]\) is a reflexive subspace of \(J\) which is not uniformly convexifiable.

**Remarks.** (i) It is easy to prove that if \(\sup_{k} t_{k} < \infty\) then \((\sum_{k=1}^{\infty} J_{k})_{i_2}\) is isomorphic to \(i_2\).

(ii) By a result of Edelstein and Mityagin [4, Lemma 5], for any \(t_{1} < t_{2} < \cdots < s_{1} < s_{2} < \cdots\), \((\sum_{k=1}^{\infty} J_{k})_{i_2}\) is isomorphic to \((\sum_{k=1}^{\infty} J_{s_{k}})_{i_2}\).

**Theorem 5.** For any subsequence \(\{e_{n}\}\) of \(\{e_{n}\}\), \([e_{n}]\) is complemented in \(J\).

**Proof.** We may assume \(\{n_{i}\}\) is a proper sequence. Let \(m_{0} = 0\) and let \(\{m_{k}\}\) be the sequence complementary to \(\{n_{i}\}\) in \(N\). Let \(X = [e_{n}]\) and let \(W\) be the closed linear space of \(A \cup B\), where \(A\) is the set of all vectors of the form \(\sum_{i=n}^{m_{i-1}} e_{j}\) with \(n_{i-1} \in \{m_{k}\}\) and \(m_{i} = \min \{m_{k} : m_{k} > n_{i-1}\}\), and \(B\) is the set of all \(e_{j}\) such that \(j, j+1\) are in \(\{m_{k}\}\). Then

\[
\|w\| < \|x + w\| \quad \text{if } x \in X \text{ and } w \in W,
\]

since \(\|w\|\) can be evaluated by using (1) with \(\{p_{i}\} \subset \{m_{k}\}\) and, for such \(\{p_{i}\}\), the sum in (1) is the same for \(x + w\) as for \(w\). Therefore, there exists a projection \(P\) of \(J\) onto \(X\) which satisfies \(P^{-1}(0) = W\) and \(\|P\| < 2\).

**Remarks.** (i) There exist complemented reflexive subspaces of \(J\) which are not uniformly convexifiable.

(ii) In Theorem 5, the subspace \(W\) is isometrically isomorphic to \(J\).

(iii) It is well known (e.g., [13, Theorem 16.8]) that a basis \(\{e_{n}\}\) in a Banach space \(X\) is unconditional if and only if that for any complementary sequences \(\{n_{i}\}\) and \(\{m_{i}\}\) in \(N\), \(X = [e_{n}] \oplus [e_{m}]\). In a sense, \(J\) has the extreme opposite property. By Corollary 3 and Theorem 5, if \(\{n_{i}\}\) is a proper sequence and \(\{m_{i}\}\)
is its complement in \( N \), then \([e_n]\) and \([e_m]\) are complemented in \( J \), but \( J \neq [e_n] \oplus [e_m] \).

**Theorem 6.** Let \( \{m_k\} \) be a proper sequence. If \( \{n_k\} \) is the sequence complementary to \( \{m_k\} \) in \( N \) and \( X = [e_n] \), then \( J \) is isomorphic to the quotient space \( J/X \).

**Proof.** As in the proof of Theorem 5, \( X \) has a complement \( W \) that is isometric to \( J \). Therefore, \( J \) is isomorphic to \( J/X \).

**Remark.** By a different method, Theorem 6 was obtained by Edelstein and Mityagin [4, Lemma 6].

2. Unconditional basic sequences. In this section, we show that \( J \) has uncountably many mutually nonequivalent unconditional basic sequences but every subsymmetric basic sequence in \( J \) is equivalent to the unit vector basis of \( l_2 \).

The following proposition is proved by Herman and Whitley [7] and is also an immediate consequence of the proof of Corollary 3.

**Proposition 7.** Let

\[
y_n = \sum_{i=p_n}^{q_n} \alpha_i e_i, \quad n = 1, 2, \ldots ,
\]

be a bounded block basic sequence of \( \{e_n\} \). If \( p_{n+1} - q_n > 1, \ n = 1, 2, \ldots , \)
then \( \{y_n\} \) is equivalent to the unit vector basis of \( l_2 \).

**Corollary 8.** \( J \) has unique subsymmetric basic sequence up to equivalence.

**Proof.** Let \( \{y_n\} \) be a subsymmetric basic sequence in \( J \). Since \( J \) does not contain a subspace isomorphic to \( l_1 \), we may assume that \( \{y_n\} \) is a block basic sequence of \( \{e_n\} \) (e.g., [1]). By Proposition 7, \( \{y_n\} \) is equivalent to the unit vector basis of \( l_2 \).

**Corollary 9.** Every bounded unconditional block basic sequence \( \{y_n\} \) of \( \{e_n\} \) is equivalent to the unit vector basis of \( l_2 \).

**Proof.** This follows immediately from Proposition 7 and the fact that, since \( \{y_n\} \) is an unconditional basic sequence, \( \sum_{n=1}^{\infty} \alpha_n y_n \) converges if and only if \( \sum_{n=1}^{\infty} \alpha_n 2^n y_n, \sum_{n=1}^{\infty} \alpha_n 2^{n-1} y_{2n-1} \) converge.

**Theorem 10.** Let \( y_n = \sum_{i=p_n}^{q_n} \alpha_i e_i, \ n = 1, 2, \ldots , \)
be a normalized block basic sequence of \( \{e_n\} \). If \( p_{n+1} - q_n > 1, \ n = 1, 2, \ldots \), then \( [y_n] \) is complemented in \( J \).

**Proof.** Let \( \{n_i\} \) denote the sequence \( p_1, \ldots , q_1, \ p_2, \ldots , q_2, \ldots \). By Corollary 3, there exists a sequence \( \{t_k\} \) in \( N \) such that \( \{e_n\} \) is equivalent to the natural basis of \( (\sum_{i=k}^{\infty} J_{t_i}) \). Let \( T \) be the isomorphism establishing the equivalence. For each \( k \in N \), choose \( f_k \in J_{t_k}^* \), \( ||f_k|| = 1 \), such that \( f_k(Ty_k) = ||Ty_k|| \). For \( x = \sum_{i=k}^{\infty} \alpha_n e_n \in [e_n] \), define...
\[ P(x) = T^{-1} \left\{ \sum_{k=1}^{\infty} \frac{1}{\|T_k\|} f_k \left( T \left( \sum_{j=p_k}^{q_k} \alpha_j e_j \right) \right) T_k \right\}. \]

From Lemma 2 and the definition of the norm in \((\Sigma_{k=1}^{\infty} J_k)_{l_2}\),
\[
\|P(x)\| < \|T\| \|T^{-1}\| \left( \sum_{k=1}^{\infty} \left\| \sum_{j=p_k}^{q_k} \alpha_j e_j \right\|^2 \right)^{1/2} < \sqrt{2} \|x\|.
\]

It follows that \(P\) is well defined. Routine calculations show that \(P\) is a bounded projection of \(E_k\) onto \(y_n\). Since \(E_k\) is complemented in \(J\) by Theorem 5, the proof is complete.

**Corollary 11.** For every infinite dimensional subspace \(E\) in \(J\) there exists a subspace \(F\) in \(E\) such that \(F\) is complemented in \(J\) and \(F\) is isomorphic to \(l_2\).

**Proof.** \(E\) contains a normalized basic sequence \(\{z_n\}\) that is equivalent to the unit vector basis of \(l_2\) (e.g., [7]). We may choose a normalized block basis \(\{y_n\}\) of \(\{e_n\}\), which satisfies the condition of Theorem 10, and a subsequence \(\{w_n\}\) of \(\{z_n\}\) such that
\[
\sum_{n=1}^{\infty} \|y_n - w_n\| < \sqrt{2}/2.
\]

By Theorem 10, there exists a projection \(P\) from \(J\) onto \(y_n\) with \(\|P\| < \sqrt{2}\). If \(\{y_n^*\}\) are the coefficient functionals of \(\{y_n\}\), then \(\|y_n^*\| = 1\) for all \(n\). Since
\[
\|P\| \sum_{n=1}^{\infty} \|y_n - w_n\| \cdot \|y_n^*\| < 1,
\]
by a result of C. Bessaga and A. Pelczynski (e.g., [13]), we conclude that \(\{w_n\}\) is complemented in \(J\).

The results of Theorems 6 and 10 also suggest that \(J\) might be a primary space. Recall that a Banach space \(X\) is primary if for any projection \(P\) on \(X\) either \(PX\) or \((I - P)X\) is isomorphic to \(X\).

Next, we use the fact [5] that \(c_0\) is finitely representable in \(J\) to show that \(J\) has uncountably many mutually nonequivalent unconditional basic sequences. Let \(l^\infty_n\) (resp. \(l^\infty_n\)) be the Banach space of all \(x = (a_1, \ldots, a_n)\) with
\[
\|x\| = \sup_i |a_i| \quad \text{(resp. } \|x\| = \sum_{i=1}^{n} |a_i| \text{)}.
\]

The following result is an immediate consequence of [5].

**Proposition 12.** Given \(\lambda > 1\), then for any positive integer \(n\) there exists an integer \(m\) such that there exists an isomorphism \(T\) from \(l^\infty_n\) into \(J_m\) such that
\[
\lambda^{-1} \|x\| < \|Tx\| < \lambda \|x\| \text{ for all } x \in l^\infty_n.
\]

**Theorem 13.** Let \(\{n_i\}\) be a sequence of positive integers. Then
(i) The Banach space \((\Sigma_{i=1}^{\infty} l^\infty_{n_i})_{l_1}\) is isomorphic to a complemented subspace of \(J\).
(ii) The Banach space \((\sum_{i=1}^{\infty} l_1^i)_{l_1}\) is isomorphic to a subspace of \(J\).

**Proof.** (i) By Proposition 12, there exist a sequence \(\{m_i\}\) and isomorphisms \(T_i\) from \(l_1^{m_i}\) into \(J_{m_i}\) with sup \(||T_i|| < \sqrt{2}\) and sup \(||T_i^{-1}|| < \sqrt{2}\). Hence \((\sum_{i=1}^{\infty} l_1^{m_i})_{l_1}\) is isomorphic to a subspace of \((\sum_{i=1}^{\infty} J_{m_i})_{l_1}\). Furthermore, for each \(i = 1, 2, \ldots\), there exists a projection \(P_i\) from \(J_{m_i}\) onto \(J_{m_i}(l_1^{m_i})\) with \(||P_i|| < 2\) and by Theorem 5, \((\sum_{i=1}^{\infty} J_{m_i})_{l_1}\) is isomorphic to a complemented subspace of \(J\). Thus \((\sum_{i=1}^{\infty} l_1^{m_i})_{l_1}\) is isomorphic to a complemented subspace of \(J\).

(ii) This follows immediately from (i) and the fact that \(l_1\) is finitely representable in \(c_0\).

**Corollary 14.** There are uncountably many mutually nonequivalent unconditional basic sequences in \(J\).

3. **Block basic sequences of \(\{e_n\}\) with constant coefficients.** It is well known that if \(\{x_n\}\) is a symmetric basis of a Banach space \(X\), then for every block basic sequence \(\{y_n\}\), the “averaging projection” is a bounded projection from \(X\) onto \([y_n]\). Although \(\{e_n\}\) is conditional, we show in this section, by making use of the “averaging projection”, that every block basic sequence of \(\{e_n\}\) with constant coefficients spans a complemented subspace in \(J\). The only other known conditional bases in Banach spaces with this property seem to be the conditional bases in \(l_2\).

The following simple lemma follows immediately from the definition of the norm in \(J\).

**Lemma 15.** For any \(p_1 < p_2 < \cdots < p_{n+1}\), let

\[
y_i = \sum_{k=p_{i+1}}^{p_i} e_k, \quad i = 1, 2, \ldots, n.
\]

Then

\[
\left\| \sum_{i=1}^{n} \alpha_i e_i \right\| < \left\| \sum_{i=1}^{n} \alpha_i y_i \right\| < \sqrt{2} \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|
\]

for all real numbers \(\alpha_1, \ldots, \alpha_n\).

**Theorem 16.** Suppose \(p_1 < q_1 < p_2 < q_2 < \cdots\). Let \(y_n = \sum_{i=p_n}^{q_n} e_i, \ n = 1, 2, \ldots\). Then

(i) If \(p_{n+1} = q_n + 1\) for all except finitely many \(n\), \([y_n]\) is isomorphic to \(J\).

(ii) If \(p_{n+1} > q_n + 1\) for infinitely many \(n\), there exists a sequence of positive integers \(\{t_k\}\) such that \([y_n]\) is isomorphic to \((\sum_{k=1}^{\infty} J_{t_k})_{l_1}\).

(iii) \([y_n]\) is complemented in \(J\).

**Proof.** (i) If there exists \(m \in N\) such that \(p_{n+1} = q_n + 1\) for all \(n > m\), then \([y_n]\) is isomorphic to \(J\) by Lemma 15.

(ii) Let \(\{n_k\}\) be the sequence of all positive integers \(n\) such that \(p_{n+1} > q_n + 1\), and let \(n_0 = 0\). Let \(t_k = n_k - n_{k-1}, \ k = 1, 2, \ldots\). For each fixed \(k\), by Lemma 15, there exists an isomorphism \(T_k\) from \(J_{t_k}\) onto \([y_{n_{k-1}+1, n_k}]\).
\[ y_{n-1} + 2, \ldots, y_n \] such that \( \|x\| \leq \|T_k x\| \leq 2\|x\| \) for all \( x \in J_k \). Using the same argument as the proof of Theorem 1, we conclude that \([y_n]\) is isomorphic to \((\sum_{k=1}^{\infty} J_k)_{l_1}\).

(iii) As in the proof of Theorem 5, there is a norm one projection \( P_1 \) of \( J \) onto the closed linear span \( W \) of

\[ [y_n] \cup \{e_i : q_k < i < p_k+1 \text{ for some } k\}. \]

Since \( W \) is isometric to \( J \), via an isometry which maps the \( e_i \)'s of \( J \) onto the \( y_n \)'s and \( e_i \)'s (\( q_k < i < p_k+1 \)) of \( W \), it follows by the proof of Theorem 5, that there is a projection \( P_2 \) of \( W \) onto \([y_n]\) with \( \|P_2\| < 2 \). Thus there is a projection \( P \) of \( J \) onto \([y_n]\) with \( \|P\| < 2 \).

**Remark.** By a result of Edelstein and Mityagin [4, Lemma 5], if \( \limsup_k t_k = \infty \), then \((\sum_{k=1}^{\infty} J_k)_{l_1}\) is isomorphic to \((\sum_{n=1}^{\infty} J_n)_{l_1}\).

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**References**


**Department of Mathematics, The University of Alabama, Huntsville, Alabama 35807**

**Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242**

**Department of Mathematics, Kent State University, Kent, Ohio 44242**

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