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ESSENTIALLY HERMITIAN OPERATORS ON l_1 ARE COMPACT PERTURBATIONS OF HERMITIANS

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ABSTRACT. In this paper, we present a solution to one case of a problem of F. F. Bonsall; namely, that every essentially Hermitian operator on l_1 is a compact perturbation of a Hermitian operator.

1. Introduction. Let $B(X)$ and $C(X)$ denote, respectively, the algebras of bounded and compact linear operators on a Banach space X. The quotient algebra $B(X)/C(X)$, called the Calkin algebra, has been the object of much study lately, especially for the case when X is a complex separable Hilbert space H. Of primary interest is the problem of which properties of a residue class modulo $C(X)$ can actually be exhibited by a representative of that class. For example, it is well known that for each $T \in B(H)$, there is a $K \in C(H)$ such that $\|T + K\| = \|T\|_e$. In [6] Stampfli shows that for each $T \in B(H)$ there is a $K \in C(H)$ such that the spectrum of $T + K$ is the same as the essential spectrum of T. In [5] it is shown that if $X = l_p$, $1 < p < \infty$, and if the essential numerical range of $T \in B(X)$ contains interior points, then there is a $K \in C(X)$ such that the numerical range of $T + K$ is the same as the essential numerical range of T. In case $X = H$, a Hilbert space, the condition that the essential numerical range contain interior points can be dropped. The present paper arose out of an attempt to establish the same result for $X = l_p$, $p \neq 2$. It can be dropped in Hilbert space because every essentially Hermitian operator can be written as a compact perturbation of a Hermitian operator. However, in [1], F. F. Bonsall asks if every essentially Hermitian operator in $B(l_p)$, $p \neq 2$, is a compact perturbation of a Hermitian operator. Hence we must work on this problem first. In this paper, we solve one case of Bonsall's question by showing that every essentially Hermitian operator on l_1 is a compact perturbation of a Hermitian operator.

2. Definitions and notation. The notation is that used by Bonsall and Duncan [2], [3]. Let $\Pi = \{(x, f) \in X \times X^*: \|x\| = \|f\| = 1, f(x) = 1\}$. The spatial numerical range of $T \in B(X)$ is the set $V(T) = \{f(Tx): (x, f) \in \Pi\}$.

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The algebra numerical range of T is the set $V(B(X), T) = \{s(T): s$ is a normalized state in $B(X)^*\}$. The essential numerical range of T is the algebra numerical range of the residue class $T + C(X)$ in the Calkin algebra $B(X)/C(X)$. It can be shown that $V(B(X), T)$ is the closed convex hull of $V(T)$. Hence $V(T) \subseteq R$ if and only if $V(B(X), T) \subseteq R$. Such operators are called Hermitian. If the essential numerical range, denoted by Vess(T), is contained in R, T is called essentially Hermitian. If T is a compact perturbation of a Hermitian operator, then T is called almost Hermitian.

3. Preliminary facts. A linear operator T on l_1 can be represented by an infinite matrix $\{a_{jk}\}$, $j, k = 1, 2, \ldots$. T is bounded if and only if $\sup_k \Sigma_{j=1}^{\infty} |a_{jk}| < \infty$ and T is compact if and only if $\lim_{n\to\infty} \sup_k \Sigma_{j=n}^{\infty} |a_{jk}| = 0$. See [7, p. 278].

Following [1], let $R(T)$ be the operator defined by the matrix $\{b_{jk}\}$, where $b_{jk} = (\operatorname{Re} a_{jk})\delta_{jk}$. Then T is almost Hermitian if and only if $T - R(T)$ is compact.

A complex number z belongs to Vess(T) if and only if there is a state s in $B(X)^*$ which annihilates $C(X)$ such that $s(T) = z$. See [3, p. 127].

4. Theorem. *If $T \in B(l_1)$ is essentially Hermitian, then T is almost Hermitian.*

Proof. Let glim denote a generalized Banach limit as described in [4, p. 856]. If (x_n, f_n) is a sequence in Π such that the first n coordinates of x_n and f_n are 0, then $\operatorname{glim}_n \langle f_n, Cx_n\rangle = \lim_n \langle f_n, Cx_n\rangle = 0$ for each compact operator C. It follows that for an arbitrary operator T, $\operatorname{glim}_n\langle f_n, Tx_n\rangle$ is a number in Vess(T).

Now suppose T is not almost Hermitian. Then $T - R(T)$ is not compact. But $T - R(T)$ agrees with T except on the diagonal, and the diagonal of $T - R(T)$ is pure imaginary. By the noncompactness of $T - R(T)$, we therefore have $\lim_{n\to\infty} \sup_k \Sigma_{\gamma=n}^{\infty} |a_{jk}| > 0$. But since $T - R(T)$ is bounded, we also have $\lim_{n\to\infty} \Sigma_{j=n}^{\infty} |a_{jk}| = 0$ for each k. It follows that there exists an $\varepsilon > 0$ such that for every n, there is a $k_n \geqslant n$ such that $\Sigma_{j=n; j\neq k_n}^{\infty} |a_{jk_n}| + |\operatorname{Im} a_{k_nk_n}| \geqslant \varepsilon$. If possible, always choose k_n such that $\operatorname{Im} a_{k_nk_n} \geqslant 0$. If this is not possible, then consider $-T$ rather than T. Hence, without loss of generality, we assume $\operatorname{Im} a_{k_nk_n} \geqslant 0$.

Now let x_n be the sequence whose k_n coordinate is i (the imaginary unit), and whose remaining coordinates are 0. Let $\theta_{jn} = \arg a_{jk_n}$, and let f_n be the sequence whose first n coordinates are 0, whose k_n coordinate is $-i$, and whose remaining coordinates in order are $\exp(-i\theta_{jn})$, $j = n + 1, \ldots, j \neq k_n$. Then $(x_n, f_n) \in \Pi$ for each n, and

$$\langle f_n, (T - R(T))x_n\rangle = \left[\sum_{\substack{j=n \\ j\neq k_n}}^{\infty} |a_{jk_n}| + |\operatorname{Im} a_{k_nk_n}|\right] i.$$

Hence $\operatorname{glim}_{n\to\infty}\langle f_n, (T - R(T))x_n\rangle$ is a pure imaginary number with imaginary part exceeding ε. Hence $T - R(T)$ is not essentially Hermitian. Since $R(T)$ is Hermitian, it follows that T is not essentially Hermitian, which is a contradiction.

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