PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 67, Number 2, December 1977

AFFINE COMPLETE ORTHOLATTICES

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ABSTRACT. Every finite orthomodular lattice is affine complete.

In universal algebra the concept of the polynomial ring has been generalized to the concept of polynomial algebra $A[x_1, \ldots, x_n]$ of a given algebra A in a variety [4]. By every polynomial $\psi(x_1, \ldots, x_n) \in A[x_1, \ldots, x_n]$ one can define a function ψ: $A^n \to A$ such that every element $(a_1, \ldots, a_n) \in A^n$ is mapped to $\psi(a_1, \ldots, a_n)$. These functions are called polynomial functions of A. Similarly, as in the variety of rings, the question of which functions f: $A^n \to A$ are polynomial functions of A is of major interest.

The function f: $A^n \to A$ is called compatible if for every congruence relation θ on the algebra A, $a_1 \theta b_1, \ldots, a_n \theta b_n$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ imply $f(a_1, \ldots, a_n) \theta f(a_1, \ldots, a_n)$. Let us notice that every polynomial function is compatible [4]. The algebra A is called polynomially complete if every function of A is a polynomial function of A. In this case A is simple and finite [4]. The algebra A is called affine complete if every compatible function is a polynomial function of A [6], [8]. A lot of examples can be found in [8] where it is also mentioned that finite Boolean algebras are affine complete. This result is essentially due to Grätzer [3] and it is our purpose to extend it to orthomodular lattices. In order to study polynomial functions on ortholattices we shall do it within a more general concept.

DEFINITION. The algebra $\mathfrak{L} = (L; \wedge, \vee, ^\backprime)$ is called a polarity lattice if $\mathfrak{L}$ is a lattice concerning the operations $\wedge$, $\vee$ and if the unary operation $^\backprime$ has the properties

(1) $(x^\backprime)^\backprime = x$,

(2) $(x \vee y)^\backprime = x^\backprime \wedge y^\backprime$.

It is clear that every ortholattice and every Boolean algebra are polarity lattices. We also have to use some results on order-polynomial-complete lattices [7]. The lattice V is order-polynomial-complete if every order preserving function is a polynomial function of V. We call the polarity lattice $\mathfrak{L} = (L; \wedge, \vee, ^\backprime)$ order-polynomial-complete if the underlying lattice $V = (L; \wedge, \vee)$ is order-polynomial-complete.

LEMMA. *If the finite polarity lattice L is order-polynomial-complete then L is polynomially complete.*

Received by the editors November 16, 1976.

AMS (*MOS*) *subject classifications* (1970). Primary 06A40.

PROOF. Let $\{0, \ldots, a, \ldots, 1\}$ be the set of elements of L and let us consider $f\colon L \to L$ with the decomposition of f by

$$f(x) = f_0(x) \vee \cdots \vee f_a(x) \vee \cdots \vee f_1(x)$$

where

$$f_a(x) = \begin{cases} f(a) & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}$$

For a further decomposition consider

$$f_a(x) = \psi_a(x) \wedge \rho_a(x)$$

where

$$\psi_a(x) = \begin{cases} f(a) & \text{if } x \geqslant a, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \rho_a(x) = \begin{cases} f(a) & \text{if } x \leqslant a, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that ψ_a is an order preserving function. By hypothesis ψ_a is a polynomial function. Now consider

$$\eta_a(x) = \begin{cases} f(a) & \text{if } x \geqslant a^`, \\ 0 & \text{otherwise,} \end{cases}$$

which is also order preserving and therefore a polynomial function. If we substitute $x^`$ for the variable $x$ in the polynomial function $\eta_a$, we get a polynomial function of the polarity lattice $L$ with $\eta_a(x^`) = \rho_a(x)$. Therefore f is a polynomial function of L. If the unary functions of L are polynomial functions, then the binary functions are polynomial functions, too. Consider $h\colon L^2 \to L$ and decompose

$$h(x,y) = h_{00}(x,y) \vee \cdots \vee h_{ab}(x,y) \vee \cdots \vee h_{11}(x,y),$$

where

$$h_{ab}(x,y) = \begin{cases} h(a, b) & \text{if } x = a \text{ and } y = b, \\ 0 & \text{otherwise.} \end{cases}$$

We have the further decomposition $h_{ab}(x,y) = \psi_{ab}(x) \wedge \rho_{ab}(y)$ where

$$\psi_{ab}(x) = \begin{cases} f(a, b) & \text{if } x = a, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \rho_{ab}(y) = \begin{cases} f(a, b) & \text{if } y = b, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 11.2 in [**4**] L is polynomially complete.

THEOREM 1. *The finite simple orthomodular lattice L is polynomially complete.*

PROOF. It is well known that L is relatively complemented [**1**, p. 53]. As L is finite, Corollary 2 of [**1**, p. 53] implies that every element of $L \setminus \{0\}$ is the finite join of atoms of L. Therefore L is atomistic and relatively complemented. The hypothesis for Theorem 10.14 of [**5**, p. 47] is fulfilled. As L is simple it is also irreducible. By 10.14.3 of [**5**] the atoms of L are projective to each other. That implies that the atoms of L are pseudoprojective to each other in the sense of Definition 3 of [**7**]. Now by Satz 3 of [**7**] L is order-polynomial-complete, and by the lemma L is polynomially complete.

THEOREM 2. *Every finite orthomodular lattice L is affine complete.*

PROOF. If L is a finite orthomodular lattice then L is the direct product of simple orthomodular lattices which are polynomially complete. By [**8**, 4.9 Korollar] this is a diagonal product, and by [**8**, 5.6 Korollar] L is affine complete.

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