SIMPLE CONTINUED FRACTIONS AND SPECIAL RELATIVITY

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Abstract. Let $E_0, E_1, \ldots, E_n$ be inertial frames of reference in a one dimensional relativistic universe where the speed of light is $c = \sqrt{k}$, $k$ some natural number. For $n > 1$ let $E_n$ have velocity 1 with respect to $E_{n-1}$. Let $x_n$ denote the velocity of $E_n$ with respect to $E_0$. Then only if $k = 2, 3$ or 5 will $x_n$ be a simple continued fraction convergent of $\sqrt{k}$ infinitely often.

Introduction. Let $c$ denote the speed of light, and let $E_0, E_1$ and $E_2$ be inertial frames of reference with collinear velocities. Let the velocity of $E_1$ with respect to $E_0$ be $x$, and of $E_2$ with respect to $E_1$, $y$. The theory of special relativity then gives the velocity of $E_2$ with respect to $E_0$ as $(x + y)/(1 + xy - 2)$, which we denote $x \oplus_c y$.

Let units of measurement be chosen so that $c^2$ is an integer. Given $x_0$, let $x_n$ denote the sum $x_0 \oplus_c 1 \oplus_c 1 \cdots \oplus_c 1$ ($n$ additions of 1), so that $x_n = x_{n-1} \oplus_c 1$ if $n > 1$. If $x_0$ is rational, the sequence $(x_n)$ will be a sequence of rational numbers converging to the square root of the integer $c^2$.

We ask whether $x_n$ is infinitely often a simple continued fraction convergent of $c$. K. B. Stolarsky noted that if $x_0 = 1$ and $c = \sqrt{2}$ or $\sqrt{3}$, this is the case, and posed the more general problem. First let us note one more example. If $c = \sqrt{5}$ and $x_0 = 1$ then $x_{6n+2}$ is the $2n$th convergent of $\sqrt{5}$. In what follows we show why there are no other examples. I wish to thank Professor Freeman Dyson for his helpful comments.

Theorem. If $c^2 = k$ is a nonsquare integer $> 5$ and if $x_0 \in \mathbb{Q}$ then there are at most finitely many $n$ such that $x_n$ is a continued fraction convergent of $\sqrt{k}$.

Proof. Let $a = a_0 + a_1 \sqrt{k}$ be such that $a_0, a_1 \in \mathbb{Z}$ and $x_0 = \sqrt{k} (a - \bar{a})/(a + \bar{a})$, where conjugation is in the field $\mathbb{Q}(\sqrt{k})$. Let $a = k + \sqrt{k}$.

Lemma. If $x = \sqrt{k} (b - \bar{b})/(b + \bar{b})$, $b \in \mathbb{Q}(\sqrt{k})$, then $x \oplus_c 1 = \sqrt{k} (b\alpha - \bar{b}\bar{\alpha})/(b\alpha + \bar{b}\bar{\alpha})$.

Proof. Let $b = b_0 + b_1 \sqrt{k}$ with $b_0, b_1 \in \mathbb{Q}$. Then

$$x \oplus_c 1 = (b_1k/b_0 + 1)/(1 + b_1/b_0) = (b_1k + b_0)/(b_1 + b_0).$$
Likewise
\[
\sqrt{k} \frac{(b\alpha - \bar{b}\alpha)}{(b\alpha + \bar{b}\alpha)} = \sqrt{k} \left( \frac{(b_1k + b_0)\sqrt{k}}{(b_0k + b_1k)} \right)
= \frac{(b_1k + b_0)}{(b_1 + b_0)}. \quad \square
\]

**Corollary.** If \( x_0, a \) and \( \alpha \) are as above, then
\[
x_n = \sqrt{k} \frac{(aa^n - \bar{a}\alpha^n)}{(aa^n + \bar{a}\alpha^n)}.
\]

**Proof.** Take \( b = aa^{n-1} \). Then \( \bar{b} = \bar{a}\alpha^{n-1} \) and the induction step follows from the Lemma above. And by assumption the claim holds when \( n = 0 \). \( \square \)

Recall that for each nonsquare \( k \) there exists an \( M \) such that if \( (d_1, d_2) = 1 \) and \( d_1/d_2 \) is a continued fraction convergent of \( \sqrt{k} \), then \( |d_1^2 - kd_2^2| < M \).

Now suppose that in \( \mathbb{Q}(\sqrt{k}) \) there is some prime ideal \( p \) which divides \( \bar{a}/\alpha \) to a positive power. This happens if and only if \( \alpha/\bar{a} \) is not an integer of \( \mathbb{Q}(\sqrt{k}) \). If for some \( n \), \( x_n = (d_1/d_2) \) is a continued fraction convergent of \( \sqrt{k} \) then by the Corollary,
\[
\sqrt{k} \frac{(aa^n - \bar{a}\alpha^n)}{(aa^n + \bar{a}\alpha^n)} = d_1/d_2
\]
so that
\[
k(aa^n - \bar{a}\alpha^n)^2d_2^2 = (aa^n + \bar{a}\alpha^n)^2d_1^2 = 0,
\]
and
\[
k(a - \bar{a}(\alpha/\alpha^n)^2d_2^2 = (a + \bar{a}(\alpha/\alpha^n)^2d_1^2 = 0.
\]

Since \( p^n \) divides \( (\alpha/\alpha)^n \), \( p^n \) divides \( a^2(\bar{d}_2^2 - d_1^2) \). Since \( kd_2^2 - d_1^2 \) is a nonzero integer with absolute value \( < M \) it contains only finitely many powers of \( p \). Therefore there can be only finitely many \( n \) for which \( x_n \) is a continued fraction convergent of \( \sqrt{k} \).

To complete the proof we note that \( \alpha/\bar{a} \) is an integer of \( \mathbb{Q}(\sqrt{k}) \) if and only if the polynomial
\[
(x - \alpha/\bar{a})(x - \bar{a}/\alpha) = x^2 - 2(k + 1)x/ (k - 1) + 1
\]
has integer coefficients, and that this occurs exactly when \( k = 2, 3 \) or 5. \( \square \)

**Bibliography**


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