ON CONTRACTIONS SATISFYING $\text{Alg } T = \{ T \}'$

PEI YUAN WU

ABSTRACT. For a bounded linear operator $T$ on a Hilbert space let $\{ T \}'$, $\{ T \}''$ and $\text{Alg } T$ denote the commutant, the double commutant and the weakly closed algebra generated by $T$ and 1, respectively. Assume that $T$ is a completely nonunitary contraction with a scalar-valued characteristic function $\psi(\lambda)$. In this note we prove the equivalence of the following conditions: (i) $|\psi(e^{i\theta})| = 1$ on a set of positive Lebesgue measure; (ii) $\text{Alg } T = \{ T \}'$; (iii) every invariant subspace for $T$ is hyperinvariant. This generalizes the well-known fact that compressions of the shift satisfy $\text{Alg } T = \{ T \}'$.

For an arbitrary operator $T$ on a Hilbert space it is easily seen that the inclusions $\text{Alg } T \subseteq \{ T \}' \subseteq \{ T \}''$ hold. Let $H^2$ be the usual Hardy space and let $\psi$ be a scalar-valued inner function. Consider the compression of the shift $T$ defined on the space $H^2 \ominus \psi H^2$ by

$$(Tf)(e^{i\theta}) = P[e^{i\theta}f(e^{i\theta})] \quad \text{for } f \in H^2 \ominus \psi H^2,$$

where $P$ denotes the (orthogonal) projection onto the space $H^2 \ominus \psi H^2$. It was shown by Sarason [3] that $\text{Alg } T = \{ T \}'$. (In fact, he showed more than this. He proved that every operator in $\{ T \}'$ is of the form $U(T)$ for some $U \in H^\infty$.) Note that here $T$ is a completely nonunitary (c.n.u.) contraction whose characteristic function $\psi$ is scalar-valued and satisfies $|\psi(e^{i\theta})| = 1$ a.e. In this note we give necessary and sufficient conditions that a c.n.u. contraction with a scalar-valued characteristic function satisfy $\text{Alg } T = \{ T \}'$. Indeed, we want to prove

**Theorem.** Let $T$ be a c.n.u. contraction with a scalar-valued characteristic function $\psi$. Then the following conditions are equivalent to each other:

(i) $|\psi(e^{i\theta})| = 1$ on a set of positive Lebesgue measure;
(ii) $\text{Alg } T = \{ T \}'$;
(iii) every invariant subspace for $T$ is hyperinvariant.

Thus Sarason's result follows from the implication (i) $\Rightarrow$ (ii) of our Theorem. It is interesting to contrast our result with the fact, due to Sz.-Nagy and Foiaş [6], that a c.n.u. contraction $T$ with the scalar-valued characteristic function $\psi$ satisfies $\{ T \}'' = \{ T \}'$ if and only if $\psi(\lambda) \equiv 0$. Note also that

Received by the editors January 17, 1977 and, in revised form, March 11, 1977.

AMS (MOS) subject classifications (1970). Primary 47A45; Secondary 47A15, 47C05.

Key words and phrases. Completely nonunitary contractions, characteristic functions, invariant subspaces, hyperinvariant subspaces, commutants, algebras of operators.

The author acknowledges financial support from National Science Council of Taiwan.

© American Mathematical Society 1978
whether (ii) and (iii) are equivalent for an arbitrary operator \( T \) is still an open question (cf. [1]).

In the proof of our Theorem we will extensively use the functional model for c.n.u. contractions. The readers are referred to [5] for the basic definitions and terminologies. Throughout this note results from [5] will be used without specific mentioning.

Let \( T \) be a c.n.u. contraction with the scalar-valued characteristic function \( \psi \). Consider the functional model for \( T \), that is, consider \( T \) being defined on the space \( H \equiv [H^2 \oplus \Delta L^2] \oplus \{\psi w \oplus \Delta w : w \in H^2\} \) by

\[
T(f \oplus g) = P(e^{\imath f} \oplus e^{\imath g}) \quad \text{for} \ f \oplus g \in H,
\]

where \( \Delta = (1 - |\psi|^2)^{1/2} \) and \( P \) denotes the (orthogonal) projection onto \( H \).

Let \( \text{Lat } T \) denote the lattice of invariant subspaces for \( T \), and let \( T^{(n)} \) denote the operator \( T \oplus \cdots \oplus T \) acting on the space \( H \oplus \cdots \oplus H \). Note that the characteristic function of \( T^{(n)} \) is the \( n \times n \) matrix-valued function

\[
\Phi = \begin{bmatrix}
\psi & 0 \\
\vdots & \ddots \\
0 & \psi
\end{bmatrix}.
\]

Let \( K \in \text{Lat } T^{(n)} \) and let \( \Phi = \Phi_2 \Phi_1 \) be the corresponding regular factorization. We first prove the following

**Lemma.** If \( |\psi(e^{it})| = 1 \) on a set of positive Lebesgue measure, then \( \Phi_1 \) and \( \Phi_2 \) are \( n \times n \) matrix-valued functions.

**Proof.** Assume that \( \Phi_1 \) and \( \Phi_2 \) are, respectively, \( m \times n \) and \( n \times m \) matrix-valued functions. Let

\[
\Delta(e^{it}) = (1 - \Phi(e^{it})^* \Phi(e^{it}))^{1/2}
\]

and

\[
\Delta_j(e^{it}) = (1 - \Phi_j(e^{it})^* \Phi_j(e^{it}))^{1/2}, \quad j = 1, 2.
\]

Let \( \delta(e^{it}) = \dim \Delta(e^{it}) \mathbb{C}^n \), \( \delta_1(e^{it}) = \dim \Delta_1(e^{it}) \mathbb{C}^n \), and \( \delta_2(e^{it}) = \dim \Delta_2(e^{it}) \mathbb{C}^n \), where \( \mathbb{C} \) denotes the complex plane. Since \( \Phi = \Phi_2 \Phi_1 \) is a regular factorization, we have

\[
\delta(e^{it}) = \delta_1(e^{it}) + \delta_2(e^{it}) \quad \text{a.e.}
\]

(cf. [5, Proposition VII. 3.3]). Since \( |\psi(e^{it})| = 1 \) on a set of positive Lebesgue measure, say \( \alpha \), it follows that \( \Delta(e^{it}) = 0 \) on \( \alpha \). Hence \( \delta(e^{it}) = 0 \) on \( \alpha \). If \( m > n \) then \( \Phi_2(e^{it}) \) cannot be isometric from \( \mathbb{C}^m \) to \( \mathbb{C}^n \). Thus \( \delta_2(e^{it}) > 0 \) a.e., which contradicts (1). On the other hand, if \( m < n \), then \( \Phi_1(e^{it}) \) cannot be isometric from \( \mathbb{C}^n \) to \( \mathbb{C}^m \). Then \( \delta_1(e^{it}) > 0 \) a.e. and we also have a contradiction. This proves that \( m = n \).

**Proof of the Theorem.** If \( \psi \equiv 0 \), then, by the previously mentioned result of Sz.-Nagy and Foias [6], it is easily seen that none of the three conditions is
satisfied. Hence we may assume hereafter that $\psi \neq 0$.

(i) $\Rightarrow$ (ii). Let $S$ be an operator in $\{ T \}'$. To show that $S \in \text{Alg } T$ it suffices to show that $\text{Lat } T^{(n)} \subseteq \text{Lat } S^{(n)}$ for all $n \geq 1$ (cf. [2, Theorem 7.1]). Let $K \subseteq \text{Lat } T^{(n)}$ and $\Phi = \Phi_2 \Phi_1$ be the corresponding regular factorization. As proved in the Lemma, $\Phi_1$ and $\Phi_2$ are $n \times n$ matrix-valued functions. In the functional model of $T^{(n)}$,

\[ K = \left\{ \Phi_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(C^n), v \in \Delta_1 L^2(C^n) \right\} \]

\[ \oplus \left\{ \Phi_2 w \oplus \Delta w : w \in H^2(C^n) \right\}, \]

where $Z$ denotes the unitary operator from $\Delta L^2(C^n)$ to $\Delta_2 L^2(C^n) \oplus \Delta_1 L^2(C^n)$ defined by

\[ Z(\Delta v) = \Delta_2 \Phi_1 v \oplus \Delta_1 v, \quad v \in L^2(C^n). \]

Let $\Phi_2 u \oplus t$ be an element in $K$, where $u = (u_i) \in H^2(C^n)$ and $t = (t_i) \in \Delta L^2(C^n)$ satisfy $Z(t) = \Delta_2 u \oplus v$ for some $v = (v_i) \in \Delta_1 L^2(C^n)$.

Here we use the symbol $(\ )_i$ to denote the components of a vector. We want to show that $S^{(n)}(\Phi_2 u \oplus t) \in K$. Note that $S$ is of the form

\[ S = P \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \]

where $A \in H^\infty$ and $B, C \in L^\infty$ satisfy $B\psi + CA = \Delta A$ a.e. (cf. [6]). Assume that $\Phi_1 = (\xi_j)$ and $\Phi_2 = (\psi_j)$. Since

\[ \Phi_2 u \oplus t = \left( \sum_{j=1}^{n} \psi_j u_j \right) \oplus (t_i)_i \]

we have

\[ S^{(n)}(\Phi_2 u \oplus t) = \left( \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{n} \psi_j u_j \\ t_i \\ \vdots \end{pmatrix} \right)_i \]

\[ = \left( \begin{pmatrix} A \sum_{j=1}^{n} \psi_j u_j \\ B \sum_{j=1}^{n} \psi_j u_j + Ct_i \end{pmatrix} \right)_i \]

\[ = \left( \begin{pmatrix} A \sum_{j=1}^{n} \psi_j u_j - \psi_i \\ B \sum_{j=1}^{n} \psi_j u_j + Ct_i - \Delta w_i \end{pmatrix} \right)_i, \]

for some $w_i \in H^2$, $i = 1, 2, \ldots, n$. Since $\Phi = \Phi_2 \Phi_1$, we have
ON CONTRACTIONS SATISFYING Alg $T = \{ T \}'$

$$
\sum_{k=1}^{n} \psi_{ik} \xi_{kj} = \begin{cases} 
\psi, & \text{if } j = i, \\
0, & \text{otherwise},
\end{cases} \quad i, j = 1, \ldots, n.
$$

Using Cramer's rule to solve this system of equations for $\psi_{ik}$, we obtain

$$(\text{det } \Phi_1) \psi_{ik} = \psi \eta_{ik} \quad i, k = 1, \ldots, n,$$

where $\eta_{ik}$ is the determinant, multiplied by $(-1)^{i+k}$, of the matrix obtained from $\Phi_1$ by deleting its $i$th column and $k$th row. It follows that

$$(\text{det } \Phi_1) B \sum_{j=1}^{n} \psi_{ij} u_j = B \sum_{j=1}^{n} \psi \eta_{ij} u_j = \Delta(A - C) \sum_{j=1}^{n} \eta_{ij} u_j.$$

Hence $$(\text{det } \Phi_1) B (\sum_{j=1}^{n} \psi_{ij} u_j)$$ is an element of $\overline{L^2(\mathbb{C}^n)}$. Thus we have

$$Z \left( (\text{det } \Phi_1) \left( \sum_{j=1}^{n} \psi_{ij} u_j \right) \right) = Z \left[ \Delta(A - C) \left( \sum_{j=1}^{n} \eta_{ij} u_j \right) \right].$$

$$= \left[ \Delta_2 \Phi_1(A - C) \left( \sum_{j=1}^{n} \eta_{ij} u_j \right) \right] \oplus \left[ \Delta_1(A - C) \left( \sum_{j=1}^{n} \eta_{ij} u_j \right) \right].$$

Since

$$\sum_{k=1}^{n} \xi_{ik} \left( \sum_{j=1}^{n} \eta_{kj} u_j \right) = \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \xi_{ik} \eta_{kj} \right) u_j$$

$$= \sum_{j=1}^{n} (\text{det } \Phi_1) \delta_{ij} u_j \quad (\delta_{ij} \text{ the Kronecker } \delta) = (\text{det } \Phi_1) u_i,$$

the above becomes

$$\left[ \Delta_2(A - C)((\text{det } \Phi_1) u_i) \right] \oplus \left[ \Delta_1(A - C) \left( \sum_{j=1}^{n} \eta_{ij} u_j \right) \right].$$

(3)

$$= \left[ \Delta_2(A - C)(\text{det } \Phi_1) u \right] \oplus \left[ \Delta_1(A - C) \left( \sum_{j=1}^{n} \eta_{ij} u_j \right) \right].$$

On the other hand,

$$Z \left[ (\text{det } \Phi_1) B \left( \sum_{j=1}^{n} \psi_{ij} u_j \right) \right] = (\text{det } \Phi_1) Z \left( B \sum_{j=1}^{n} \psi_{ij} u_j \right),$$

(4)

$$= (\text{det } \Phi_1)(X \oplus Y),$$

say, for some element $X \oplus Y$ in $\overline{\Delta_2 L^2(\mathbb{C}^n)} \oplus \overline{\Delta_1 L^2(\mathbb{C}^n)}$. Equating the first components in (3) and (4) we obtain
Since $\psi \equiv 0$, we have $\det \Phi \equiv 0$, and hence $\det \Phi_i \equiv 0$. By the F. and M. Riesz theorem, (5) yields that $\Delta_2(A - C)u = X$. Thus

\[
Z \left( B \sum_{j=1}^{n} \psi_j u_j + Ct_i \right) = Z \left( B \sum_{j=1}^{n} \psi_j u_j \right) + Z ((Ct_i)_i) = (X \oplus Y) + Z (Ci) = [\Delta_2(A - C)u \oplus Y] + C (\Delta_2u \oplus v) = \Delta_2 Au \oplus (Y + Cv).
\]

Hence (2) can be written as

\[
S^{(n)}(\Phi_2u \oplus i) = [\Phi_2 Au \oplus Z^{-1} [\Delta_2 Au \oplus (Y + Cv)]] - (\Phi w \oplus \Delta w),
\]

where $w = (w_j)_j \in H^2(C^n)$. This shows that $S^{(n)}(\Phi_2u \oplus i) \in K$ as asserted and completes the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). Assume $|\psi(e^{it})| < 1$ a.e. It was proved in [7] that the hyperinvariant subspaces for $T$ are of the form \{ $f \oplus g \in H$: $-\Delta f + \psi g \in L^2(E)$ and $f \in \mathcal{H}^2$ \}, where $E$ is a measurable subset of the unit circle and $I$ is an inner divisor of $\psi$, where $\psi$ denotes the inner factor of $\psi$. By Proposition 7.2 of [4], invariant subspaces of this form are precisely those arising from scalar regular factorizations of $\psi$. However, since $|\psi(e^{it})| < 1$ a.e., it is known [5, p. 301] that nontrivial vector regular factorizations of $\psi$ exist. By the uniqueness of the correspondence between regular factorizations of $\psi$ and invariant subspaces for $T$, the invariant subspace corresponding to any such vector regular factorization of $\psi$ cannot arise from a scalar regular factorization, and hence is not hyperinvariant. Thus we obtain a contradiction of (iii) and complete the proof.

**Corollary.** Let $T$ be a c.n.u. contraction with a scalar-valued inner characteristic function. Then $\text{Alg } T = \{ T \}'$.

We are grateful to the referee for making the proof of (iii) $\Rightarrow$ (i) of our Theorem more conceptual and less computational.

**References**


DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY, HSINCHU, TAIWAN, CHINA