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CLOSED CURVES OF CONSTANT TORSION. II[1]

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ABSTRACT. In this note we show that there exist closed regular C^3 space curves α with curvature $\kappa > 0$ and nonzero constant torsion τ whose total torsion $\int_\alpha \tau\, ds$ is arbitrarily small. In so doing, we give another proof of the existence of closed curves of nonzero constant torsion. This note shows that Conjecture 2 in [2] is incorrect since the preceding statement is equivalent to the statement that there exist closed curves of constant torsion $\tau = 1$ whose length is arbitrarily small.

Let α: $S^1 \to \mathbf{E}^3$ be a closed C^1 space curve. The *Peano direction* of α is the vector P_α defined by

$$P_\alpha = \int_{S^1} \alpha \times d\alpha,$$

where $\times$ is the usual cross-product in space. The Peano direction of α has two noteworthy properties:

(1) P_α is independent of the Cartesian coordinate system on $\mathbf{E}^3$ and, in particular, the origin.

(2) If α_z is the projection of α onto a plane perpendicular to the vector z ($\|z\| = 1$), then

$$P_{\alpha_z} = \int_{S^1} \alpha_z \times d\alpha_z$$

equals the projection of P_α onto z; $(P_\alpha \cdot z)z$.

Let O be the origin of $\mathbf{E}^3$ with respect to a Cartesian coordinate system on $\mathbf{E}^3$ and let S^2 be the unit sphere centered at O. Let α: $S^1 \to \mathbf{E}^3$ be a regular C^2 space curve with positive curvature. We may define α's *binormal indicatrix*, β: $S^1 \to S^2$, by regarding the binormal vector field along α as a curve on S^2. In [2] we give conditions for the existence of a closed regular C^3 space curve α with positive curvature and nonzero constant torsion in terms of α's tangent indicatrix σ. We now restate those conditions in terms of β.

PROPOSITION. *There exists a closed regular C^3 space curve with curvature $\kappa > 0$ and torsion $\tau = 1$ if and only if there exists a closed curve β on S^2*

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satisfying the following properties:

(A) *β is a regular C^2 curve with positive geodesic curvature.*

(B) $P_\beta = 0$.

PROOF. The Proposition follows immediately from the Proposition of [2] and the fact that the tangent indicatrix and binormal indicatrix are polars of each other under the spherical polarity; see [1] for the pertinent facts about the spherical polarity.

The curve β on S^2 in the preceding Proposition turns out to be the binormal indicatrix of the space curve α of constant torsion $\tau = 1$. Hence the length of β is the total torsion of the space curve α. We will establish the existence of closed curves of nonzero constant torsion whose total torsion is arbitrarily small by constructing closed curves on S^2 which satisfy properties (A) and (B) and whose length is arbitrarily small.

Let z be a unit vector, Z a line through O in the direction of z, and π a plane through O perpendicular to z. Let p: $S^2 \to \pi$ be projection in the z direction from S^2 onto π; note that p sends any curve β on S^2 into β_z.

If we construct a curve β on S^2 which is invariant under rotations of $2\pi/3$ radians about Z and for which $P_{\beta_z} = 0$, then $P_\beta = 0$ by symmetry and property (2) of the Peano direction. But note that β is invariant under rotations of $2\pi/3$ radians about Z if β_z lies in the open unit disk D about O in π and is invariant under such rotations. Hence we need only construct a plane curve β_z in D which is invariant under rotations of $2\pi/3$ radians about Z, or O, and for which $P_{\beta_z} = 0$ in order to get a curve β on S^2 for which $P_\beta = 0$. Simply let $\beta = p^{-1} \circ \beta_z$ where p^{-1} maps D onto a suitably chosen open hemisphere H of S^2.

Therefore let β_z be a closed regular C^2 curve in D satisfying the properties:

(a) β_z has positive curvature.

(b) β_z is invariant under rotations of $2\pi/3$ radians about O and $P_{\beta_z} = 0$.

Requiring that β_z have positive curvature does not, of course, imply that $\beta = p^{-1} \circ \beta_z$ on S^2 has positive geodesic curvature. However, p: $H \to D$ is a coordinate system on H which is almost normal at $H \cap Z$, i.e., the metric tensor $g_{ij} = \delta_{ij}$ and the Christoffel symbols $\Gamma^i_{jk} = 0$ for this coordinate system at $H \cap Z$. For $0 < r < 1$, $r\beta_z$ still has properties (a) and (b); in fact, the curvature of $r\beta_z \to \infty$ as $r \to 0$. Moreover, as $r \to 0$, $r\beta_z$ lies in smaller and smaller neighborhoods of O on which p^{-1} changes the geodesic curvature by lesser and lesser amounts. Hence for sufficiently small r, $p^{-1} \circ (r\beta_z)$ satisfies (A) and (B), provided H is chosen to be the hemisphere for which $p^{-1} \circ (r\beta_z)$ has positive geodesic curvature. Finally, note that the length of $p^{-1} \circ (r\beta_z) \to 0$ as $r \to 0$.

It is easy to construct examples of closed regular C^2 plane curves lying in an open unit disk about O satisfying (a) and (b). For example, β_z: $\mathbf{R} \to \mathbf{E}^2$ given by

$$\beta_z(\theta) = \left(\frac{1}{2}\cos\theta + (\sqrt{2}/4)\cos 2\theta, \frac{1}{2}\sin\theta - (\sqrt{2}/4)\sin 2\theta\right)$$

induces the required curve $\beta_z : S^1 \to \mathbf{E}^2$.

In fact for the given β_z we computed $\beta = p^{-1} \circ \beta_z$, and the numerically integrated $\beta \times d\beta/d\theta$ to obtain a closed curve of constant torsion. The projection of the resulting curve on the x, y and x, z planes is shown in Figures 1 and 2, respectively.

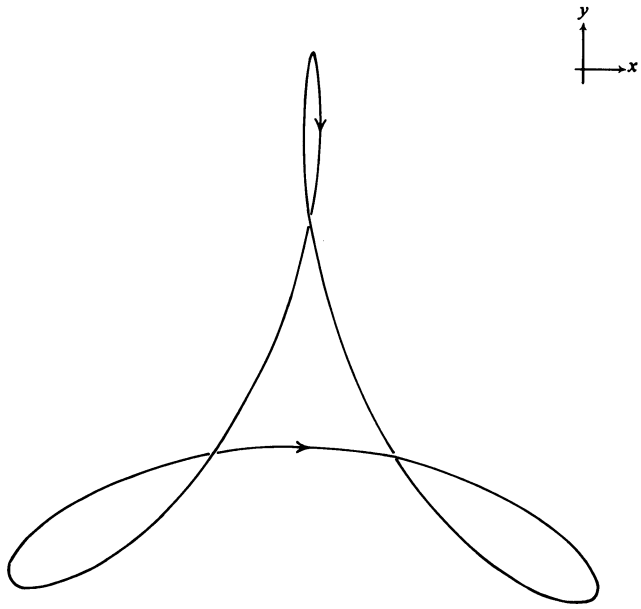

FIGURE 1

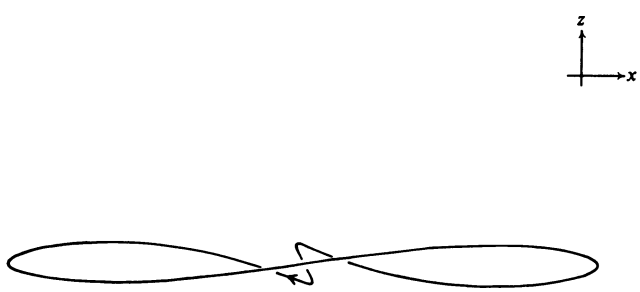

FIGURE 2

REFERENCES

1. Werner Fenchel, *On the differential geometry of closed space curves*, Bull. Amer. Math. Soc. **57** (1951), 44–54. MR **12**, 634.

2. Joel L. Weiner, *Closed curves of constant torsion*, Arch. Math. (Basel) **25** (1974), 313–317. MR **49** #11437.

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