CLOSED CURVES OF CONSTANT TORSION. II

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Abstract. In this note we show that there exist closed regular $C^3$ space curves $\alpha$ with curvature $\kappa > 0$ and nonzero constant torsion $\tau$ whose total torsion $\int_0^1 \tau \, ds$ is arbitrarily small. In so doing, we give another proof of the existence of closed curves of nonzero constant torsion. This note shows that Conjecture 2 in [2] is incorrect since the preceding statement is equivalent to the statement that there exist closed curves of constant torsion $\tau = 1$ whose length is arbitrarily small.

Let $\alpha: S^1 \to \mathbb{E}^3$ be a closed $C^1$ space curve. The Peano direction of $\alpha$ is the vector $P_\alpha$ defined by

$$P_\alpha = \int_{S^1} \alpha \times da,$$

where $\times$ is the usual cross-product in space. The Peano direction of $\alpha$ has two noteworthy properties:

1. $P_\alpha$ is independent of the Cartesian coordinate system on $\mathbb{E}^3$ and, in particular, the origin.
2. If $\alpha_z$ is the projection of $\alpha$ onto a plane perpendicular to the vector $z$ ($\|z\| = 1$), then

$$P_{\alpha_z} = \int_{S^1} \alpha_z \times da_z$$

equals the projection of $P_\alpha$ onto $z$, $(P_\alpha \cdot z)z$.

Let $O$ be the origin of $\mathbb{E}^3$ with respect to a Cartesian coordinate system on $\mathbb{E}^3$ and let $S^2$ be the unit sphere centered at $O$. Let $\alpha: S^1 \to \mathbb{E}^3$ be a regular $C^2$ space curve with positive curvature. We may define $\alpha$'s binormal indicatrix, $\beta: S^1 \to S^2$, by regarding the binormal vector field along $\alpha$ as a curve on $S^2$. In [2] we give conditions for the existence of a closed regular $C^3$ space curve $\alpha$ with positive curvature and nonzero constant torsion in terms of $\alpha$'s tangent indicatrix $\alpha$. We now restate those conditions in terms of $\beta$.

Proposition. There exists a closed regular $C^3$ space curve with curvature $\kappa > 0$ and torsion $\tau = 1$ if and only if there exists a closed curve $\beta$ on $S^2$...
satisfying the following properties:

(A) $\beta$ is a regular $C^2$ curve with positive geodesic curvature.

(B) $P_\beta = 0$.

**Proof.** The Proposition follows immediately from the Proposition of [2] and the fact that the tangent indicatrix and binormal indicatrix are polars of each other under the spherical polarity; see [1] for the pertinent facts about the spherical polarity.

The curve $\beta$ on $S^2$ in the preceding Proposition turns out to be the binormal indicatrix of the space curve $\alpha$ of constant torsion $\tau = 1$. Hence the length of $\beta$ is the total torsion of the space curve $\alpha$. We will establish the existence of closed curves of nonzero constant torsion whose total torsion is arbitrarily small by constructing closed curves on $S^2$ which satisfy properties (A) and (B) and whose length is arbitrarily small.

Let $z$ be a unit vector, $Z$ a line through $O$ in the direction of $z$, and $\pi$ a plane through $O$ perpendicular to $z$. Let $p: S^2 \to \pi$ be projection in the $z$ direction from $S^2$ onto $\pi$; note that $p$ sends any curve $\beta$ on $S^2$ into $\beta_z$.

If we construct a curve $\beta$ on $S^2$ which is invariant under rotations of $2\pi/3$ radians about $Z$ and for which $P_{\beta_z} = 0$, then $\beta_z = 0$ by symmetry and property (2) of the Peano direction. But note that $\beta$ is invariant under rotations of $2\pi/3$ radians about $Z$ if $\beta_z$ lies in the open unit disk $D$ about $O$ and is invariant under such rotations. Hence we need only construct a plane curve $\beta_z$ in $D$ which is invariant under rotations of $2\pi/3$ radians about $Z$, or $O$, and for which $P_{\beta_z} = 0$ in order to get a curve $\beta$ on $S^2$ for which $P_\beta = 0$. Simply let $\beta = p^{-1} \circ \beta_z$ where $p^{-1}$ maps $D$ onto a suitably chosen open hemisphere $H$ of $S^2$.

Therefore let $\beta_z$ be a closed regular $C^2$ curve in $D$ satisfying the properties:

(a) $\beta_z$ has positive curvature.

(b) $\beta_z$ is invariant under rotations of $2\pi/3$ radians about $O$ and $P_{\beta_z} = 0$.

Requiring that $\beta_z$ have positive curvature does not, of course, imply that $\beta = p^{-1} \circ \beta_z$ on $S^2$ has positive geodesic curvature. However, $p: H \to D$ is a coordinate system on $H$ which is almost normal at $H \cap Z$, i.e., the metric tensor $g_{ij} = \delta_{ij}$ and the Christoffel symbols $\Gamma^k_{ij} = 0$ for this coordinate system at $H \cap Z$. For $0 < r < 1$, $r\beta_z$ still has properties (a) and (b); in fact, the curvature of $r\beta_z \to \infty$ as $r \to 0$. Moreover, as $r \to 0$, $r\beta_z$ lies in smaller and smaller neighborhoods of $O$ on which $p^{-1}$ changes the geodesic curvature by lesser and lesser amounts. Hence for sufficiently small $r$, $p^{-1} \circ (r\beta_z)$ satisfies (A) and (B), provided $H$ is chosen to be the hemisphere for which $p^{-1} \circ (r\beta_z)$ has positive geodesic curvature. Finally, note that the length of $p^{-1} \circ (r\beta_z) \to 0$ as $r \to 0$.

It is easy to construct examples of closed regular $C^2$ plane curves lying in an open unit disk about $O$ satisfying (a) and (b). For example, $\beta_z: \mathbb{R} \to \mathbb{E}^2$ given by

$$\beta_z(\theta) = \left( \frac{1}{2} \cos \theta + (\sqrt{2}/4) \cos 2\theta, \frac{1}{2} \sin \theta - (\sqrt{2}/4) \sin 2\theta \right)$$
induces the required curve $\beta_z : S^1 \to E^2$.

In fact for the given $\beta_z$ we computed $\beta = p^{-1} \circ \beta_z$, and the numerically integrated $\beta \times d\beta/d\theta$ to obtain a closed curve of constant torsion. The projection of the resulting curve on the $x, y$ and $x, z$ planes is shown in Figures 1 and 2, respectively.

**Figure 1**

**Figure 2**

**References**


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