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ON PERTURBATIONS OF FREDHOLM OPERATORS IN $L_p(\mu)$-SPACES

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ABSTRACT. Answering a question of Milman, we show that a continuous linear operator T: $L_p(\mu) \to L_p(\mu)$, $1 < p < 2$, is a Fredholm perturbation iff T is strictly singular.

I. Introduction. In this note we complete the proof of the following theorem:

A continuous linear operator T: $X \to X$, $X = L_p(\mu)$, $1 \leqslant p \leqslant \infty$, is a Fredholm-perturbation iff T is strictly singular.

For $X = l_p$, $1 \leqslant p < \infty$, this was shown by Gohberg, Markus, Feldman [2], and for $X = L_1(0, 1)$ or $X = L_p(0, 1)$, $2 \leqslant p \leqslant \infty$, by Milman [7]. In [7] the remaining case $X = L_p(0, 1)$, $1 < p < 2$, was stated as a problem.

This result is contained in the following theorem:

THEOREM. *For a bounded linear operator T: $L_p(\mu) \to L_p(\mu)$, $1 < p < \infty$, the following assertions are equivalent*:

(a) *T is strictly singular.*

(b) *T is $\{l_p, l_2\}$-singular.*

(c) *T is strictly cosingular.*

(d) *T is $\{l_p, l_2\}$-cosingular.*

(e) *There is no subspace $M \subset L_p(\mu)$, isomorphic to l_p or l_2, such that M and $T(M)$ are complemented in $L_p(\mu)$ and $T|_M$ is an isomorphism.*

This theorem may be regarded as a partial analogue (for $1 < p < \infty$) to results of A. Pelczynski [9] and H. Rosenthal [12]: Let T: $L_1(\mu) \to L_1(\mu)$ or T: $L_\infty(\mu) \to L_\infty(\mu)$. Then the above-mentioned equivalences remain true if we replace $\{l_p, l_2\}$ by $\{l_1\}$ resp. $\{l_\infty\}$.

COROLLARY 1. *Let $X = L_p(\mu)$, $1 < p < \infty$. Then $S(X) = F_+(X) = F(X) = F_-(X) = C(X)$.*

COROLLARY 2. *T: $L_p(\mu) \to L_p(\mu)$, $1 < p < \infty$, is strictly singular (strictly cosingular) iff T is strictly singular (cosingular).*

Part of Corollary 1 ($S(X) = F(X)$ for $p \geqslant 2$) and one implication of Corollary 2 were already shown in [7, Theorem 1 and Corollary to

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Proposition 2]. It is well known that Corollaries 1 and 2 are false for more general Banach spaces X (see [**2**] and [**9**]). But it seems to be still an open question whether $F_+(X) = S(X)$ or $F_-(X) = C(X)$ for all Banach spaces X (see [**2**] and [**8**, Problem 20.8.6, 20.8.9]).

II. Notation. We use standard Banach space notation (see e.g. [**6**]). An "Operator" stands for a bounded linear operator in Banach spaces $X, Y, \ldots$. An operator is an isomorphism, if it is one-to-one with closed range. If (Ω, Σ, μ) is a positive measure space and $1 \leqslant p < \infty$ then $L_p(\mu)$ denotes the space of all μ-equivalence classes of p-absolutely integrable μ-measurable scalar valued functions on Ω with the norm $\|f\|_p = (\int_\Omega |f|^p \, d\mu)^{1/p}$. For a definition of "(unconditional) basic sequence", "equivalence of basic sequences" etc., see e.g. [**6**]. Let us recall some operator ideals (see [**8**] and [**11**]):

Denote by $F(X)$ $[F_+(X), F_-(X)]$ the operator ideals of all Fredholm $[\phi_+(X), \phi_-(X)]$ perturbations, which means that $T \in F(X)$ $[T \in F_+(X), T \in F_-(S)]$ iff $T + S \in \phi(X)$ $[\phi_-(X), \phi_+(X)]$ for all $S \in \phi(X)$ $[S \in \phi_+(X), S \in \phi_-(X)]$. There is some operator ideal F_0 in the sense of Pietsch, such that $F_0(X) = F(X)$ and that is (see [**8**, 5.3]).

$F_0(X, Y) = \{S\colon X \to X \mid \mathrm{Id}_X + TS \in \phi(X)$ for all $T\colon Y \to X\}$. These ideals will be compared with the following classes of operators [**3**].

Definition. Let $\mathfrak{F}$ be a class of infinite dimensional Banach spaces.

(a) An operator $T\colon X \to Y$ is said to be $\mathfrak{F}$-singular if there is no commuting diagram

$$\begin{tikzcd} X \arrow[rr, "T"] & & Y \\ & F \arrow[ul, "J_1"] \arrow[ur, "J_2"'] & \end{tikzcd}$$

where $F \in \mathfrak{F}$ and J_1, J_2 are isomorphisms.

(b) $T\colon X \to Y$ is said to be $\mathfrak{F}$-cosingular if there is no commuting diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "Q_1"'] & & Y \arrow[dl, "Q_2"] \\ & F & \end{tikzcd}$$

where $F \in \mathfrak{F}$ and Q_1, Q_2 are surjective operators.

If $\mathfrak{F}$ contains all infinite dimensional Banach spaces, then the $\mathfrak{F}$-singular operators are called strictly singular (or semicompact [**11**, p. 252]) and the $\mathfrak{F}$-cosingular operators are called strictly cosingular (or cosemicompact [**11**, p. 257]). Denote by S $[C]$ the ideals of strictly [co]singular operators. It is remarked in [**3**] that the product of two operators, one of them being

$\mathfrak{F}$-(co)singular, is $\mathfrak{F}$-(co)singular too (for arbitrary $\mathfrak{F}$). But in general the class of $\mathfrak{F}$-(co)singular operators is not an operator ideal (see **[3]**).

PROPOSITION. (a) *Let each infinite dimensional closed subspace of every $F \in \mathfrak{F}$ contain a subspace isomorphic to some element of $\mathfrak{F}$. Then the $\mathfrak{F}$-singular operators form an ideal.*

(b) *Let each infinite dimensional quotient space of every $F \in \mathfrak{F}$ have a quotient space isomorphic to some element of $\mathfrak{F}$. Then the $\mathfrak{F}$-cosingular operators form an ideal.*

PROOF. By the preceding remark we only have to show that the sum of two $\mathfrak{F}$-singular ($\mathfrak{F}$-cosingular) operators is again $\mathfrak{F}$-singular ($\mathfrak{F}$-cosingular).

(a) Let S, T: $X \to Y$ be $\mathfrak{F}$-singular and let F be a subspace of X isomorphic to some element of $\mathfrak{F}$. Then $S|_F$ and $T|_F$ are strictly singular. Otherwise there exists an infinite dimensional subspace F_1 of F, such that $S|_{F_1}$ or $T|_{F_1}$ is an isomorphism. But this contradicts the $\mathfrak{F}$-singularity of S, T since F_1 contains a subspace isomorphic to some element of $\mathfrak{F}$. Hence, by **[11**, p. 253], $(S + F)|_F$ is strictly singular and cannot be an isomorphism.

(b) Similarly, by using **[11**, C II, Corollary 6.4].

In particular, the sets of $\{l_p, l_2\}$-(co)singular operators **[6**, II.2.2] and the class of c_0-cosingular operators considered in **[3]** form operator ideals (because each quotient space of c_0 has itself a quotient space isomorphic to c_0 (see **[5**, p. 78])).

III. Proofs.

PROOF OF THE THEOREM. Since (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follows from (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (e) by duality, and trivially (a) $\Rightarrow$ (b) $\Rightarrow$ (e), it is enough to show (e) $\Rightarrow$ (a). First we show that there is no loss of generality to assume that T: $L_p(0, 1) \to L_p(0, 1)$.

Let M be a separable subspace of $L_p(\Omega, \Sigma, \mu)$ such that $T|_M$ is an isomorphism. By **[14**, Lemma 1] there exists a set $\tilde{\Omega} \in \Sigma$, a σ-subring $\tilde{\Sigma}$ of Σ restricted to $\tilde{\Omega}$ such that:

(i) $L_p(\tilde{\Omega}, \tilde{\Sigma}, \mu)$ is separable, complemented in $L_p(\Omega, \Sigma, \mu)$ and $(\tilde{\Omega}, \tilde{\Sigma}, \mu)$ is σ-finite.

(ii) $M \subset L_p(\tilde{\Omega}, \tilde{\Sigma}, \mu)$.

(iii) $L_p(\tilde{\Omega}, \tilde{\Sigma}, \mu)$ is an invariant subspace for T.

Use **[13**, §15.2] to find a space $L \supset L_p(\tilde{\Omega}, \tilde{\Sigma}, \mu)$, L isomorphic to $L_p(0, 1)$, and a projection P: $L \to L_p(\tilde{\Omega}, \tilde{\Sigma}, \mu)$. Now if $T_1 = TP$: $L \to L$ contradicts (e), then T: $L_p(\Omega, \mu) \to L_p(\Omega, \mu)$ contradicts (e).

Hence it is enough to consider operators

$$T\colon L_p(0, 1) \to L_p(0, 1), \qquad 1 < p < 2.$$

(The case $2 \leqslant p < \infty$ follows immediately from a theorem of Kadec-Pelczynski **[6**, II 3.10, 3.11]: An infinite dimensional subspace M of $L_p(0, 1)$, $2 \leqslant p < \infty$, is either isomorphic to l_2 and complemented in $L_p(0, 1)$, or it contains a subspace which is isomorphic to l_p and complemented in $L_p(0, 1)$.)

Let M be an infinite dimensional subspace of $L_p(0, 1)$ such that $T|_M$ is an isomorphism.

If M contains some subspace isomorphic to l_p, then, by a theorem of Enflo-Rosenthal, [**1**, Theorem 3.1], $T(M)$ contains even a complemented (in $L_p(0, 1)$) subspace isomorphic to l_p. But this contradicts (e).

Now we assume that M contains no subspace isomorphic l_p. By [**6**, I 1.14] there is an unconditional basic sequences $(f_n) \subset M$ with $\|f_n\| = 1$, $\|Tf_n\| > C > 0$.

We claim that for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that

$$\mu(A) < \delta(\varepsilon) \Rightarrow \sup_n \int_A |f_n|^p \, d\mu < \varepsilon^p. \tag{1}$$

Otherwise there is a subsequence (f_k') of (f_n), a sequence of measurable sets (A_k) and $D > 0$ satisfying

$$\mu(A_k) < 1/2^k, \qquad \int_{A_k} |f_k'|^p \, d\mu > D, \qquad k \in \mathbf{N}.$$

Consider the sets $B_k = \bigcup_{j=k}^{\infty} A_j$. Since

$$\int_{B_k - B_l} |f_k'|^p \, d\mu \underset{l\to\infty}{\to} \int_{B_k} |f_k'|^p \, d\mu > D$$

there is a subsequence of indices k_i such that

$$\int_{B_{k_i} - B_{k_{i+1}}} |f_{k_i}'|^p \, d\mu > D.$$

Since $(B_{k_i} - B_{k_{i+1}})_{i \in \mathbf{N}}$ are pairwise disjoint, it follows from [**4**, Lemma 2], that (f_{k_i}') is equivalent to the unit vector basis of l_p. This proves (1).

Next we construct a sequence $(g_n) \subset L_p(0, 1)$ such that

(2) $1/2 \leq \|g_k\| \leq 2$, $g_k \to^w 0$, $|g_k(t)| \leq M < \infty$ for all $t \in (0, 1)$, $\|T(g_k)\| > C/2$ and (Tg_k) is an unconditional basic sequence.

(3) Since (f_n) is bounded there is a $M_\varepsilon > 0$ for every $\varepsilon > 0$ such that $\mu(\{|f_n| > M_\varepsilon\}) < \delta(\varepsilon)$ for all integers n.

Define

$$\bar{f}_n(t) = \begin{cases} f_n(t) & \text{provided } |f_n(t)| \leq M_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

By extracting a subsequence, if necessary, we may assume that $(f_n - \bar{f}_n)$ converges weakly to some $f \in L_p(0, 1)$. Choose a measurable set B such that $\mu(\mathbf{C}B) < \delta(\varepsilon)$, $|f(t)| \leq N < \infty$ for $t \in B$. Because of (1), (3) we have $\|f_n - \bar{f}_n\| \leq \varepsilon$, $\|f\| \leq \varepsilon$. Now defining $g_n = (\bar{f}_n - f)\chi_B$, (2) is satisfied for ε small enough. (2) implies that some subsequence (g_{k_i}) of (g_k) is an unconditional basic sequence in $L_2(0, 1)$. Therefore we have for all $\alpha_1, \ldots, \alpha_r \in \mathbf{R}$:

$$\left\|\sum_{i=1}^{r} \alpha_i T(g_{k_i})\right\|_p \leqslant \|T\|\cdot\left\|\sum_{i=1}^{r} \alpha_i g_{k_i}\right\|_p$$
$$\leqslant \|T\|\cdot\left\|\sum_{i=1}^{r} \alpha_i g_{k_i}\right\|_2 \leqslant \|T\|\cdot C_1\cdot\left(\sum_{i=1}^{r}|\alpha_i|^2\right)^{1/2}.$$

On the other hand, since $T(g_{k_i})$ is an unconditional basic sequence too, it follows from a theorem of Orlicz [**6**, II 3.7], that

$$C_2\cdot\left(\sum_{i=1}^{r}|\alpha_i|^2\right)^{1/2} \leqslant \left\|\sum_{i=1}^{r} \alpha_i T(g_{k_i})\right\|_p.$$

Finally, by [**10**, Theorem 2.1], span$(T(g_{k_i}))$ contains a subspace isomorphic to l_2 and complemented in $L_p(0, 1)$. This again contradicts (e).

Proof of the Corollaries. It is well known that $S(X) \subset F_+(X) \subset F(X)$ and $C(X) \subset F_-(X) \subset F(X)$ (see [**2**] and [**11**, Chapter V]). Assume that there exists some operator $T \in F(X) - S(X)$ or $T \in F(X) - C(X)$. By the theorem, there exists some subspace M of X and a projection P: $X \to T(M)$ such that $M \hookrightarrow X \to^T X \to^P T(M)$ is an isomorphism onto $T(M)$ and belongs to $F(M, T(M))$. This contradiction proves Corollary 1. Corollary 2 follows from the theorem and the simple fact that T has property (e) iff T' has property (e).

Remark. The proof of Corollary 1 actually yields a slightly stronger result: For any proper operator ideal A in the sense of Pietsch (e.g. Id $\notin A(Y)$ for all Banach spaces Y), we have: $A(X) \subset S(X)$ for $X = L_p(\mu)$.

Added in proof. The referee kindly pointed out to me that the theorem is related to a result of H. P. Rosenthal (see Theorem 1, *Convolution by a Biased Coin*, The Altgeld Book 1975/76, University of Illinois, Functional Analysis Seminar):

Let T: $L^1(0, 1) \to L^1(0, 1)$ be a bounded linear operator so that there is a sequence (f_n) in $L^1(0, 1)$ which tends weakly to zero but (Tf_n) does not tend to zero in norm. Then there exists a subspace Y of L^1 isomorphic to l^2 with $T|Y$ an isomorphism.

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