ON PERTURBATIONS OF FREDHOLM OPERATORS IN $L_p(\mu)$-SPACES

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Abstract. Answering a question of Milman, we show that a continuous linear operator $T: L_p(\mu) \rightarrow L_p(\mu)$, $1 < p < 2$, is a Fredholm perturbation iff $T$ is strictly singular.

I. Introduction. In this note we complete the proof of the following theorem:

A continuous linear operator $T: X \rightarrow X$, $X = L_p(\mu)$, $1 < p < \infty$, is a Fredholm-perturbation iff $T$ is strictly singular.

For $X = l_p$, $1 < p < \infty$, this was shown by Gohberg, Markus, Feldman [2], and for $X = L_1(0, 1)$ or $X = L_p(0, 1)$, $2 < p < \infty$, by Milman [7]. In [7] the remaining case $X = L_p(0, 1)$, $1 < p < 2$, was stated as a problem.

This result is contained in the following theorem:

Theorem. For a bounded linear operator $T: L_p(\mu) \rightarrow L_p(\mu)$, $1 < p < \infty$, the following assertions are equivalent:

(a) $T$ is strictly singular.
(b) $T$ is $\{l_p, l_2\}$-singular.
(c) $T$ is strictly cosingular.
(d) $T$ is $\{l_p, l_2\}$-cosingular.
(e) There is no subspace $M \subset L_p(\mu)$, isomorphic to $l_p$ or $l_2$, such that $M$ and $T(M)$ are complemented in $L_p(\mu)$ and $T\mid_M$ is an isomorphism.

This theorem may be regarded as a partial analogue (for $1 < p < \infty$) to results of A. Pelczynski [9] and H. Rosenthal [12]: Let $T: L_1(\mu) \rightarrow L_1(\mu)$ or $T: L_\infty(\mu) \rightarrow L_\infty(\mu)$. Then the above-mentioned equivalences remain true if we replace $\{l_p, l_2\}$ by $\{l_1\}$ resp. $\{l_\infty\}$.

Corollary 1. Let $X = L_p(\mu)$, $1 < p < \infty$. Then $S(X) = F_+(X) = F(X) = F_-(X) = C(X)$.

Corollary 2. $T$: $L_p(\mu) \rightarrow L_p(\mu)$, $1 < p < \infty$, is strictly singular (strictly cosingular) iff $T$ is strictly singular (cosingular).

Part of Corollary 1 ($S(X) = F(X)$ for $p > 2$) and one implication of Corollary 2 were already shown in [7, Theorem 1 and Corollary to...
Proposition 2]. It is well known that Corollaries 1 and 2 are false for more
general Banach spaces X (see [2] and [9]). But it seems to be still an open
question whether \( F_+(X) = S(X) \) or \( F_-(X) = C(X) \) for all Banach spaces X
(see [2] and [8, Problem 20.8.6, 20.8.9]).

II. Notation. We use standard Banach space notation (see e.g. [6]). An
"Operator" stands for a bounded linear operator in Banach spaces \( X, Y, \ldots \).
An operator is an isomorphism, if it is one-to-one with closed range. If
\((\Omega, \Sigma, \mu)\) is a positive measure space and \( 1 < p < \infty \) then \( L_p(\mu) \) denotes the
space of all \( \mu\)-equivalence classes of \( p\)-absolutely integrable \( \mu\)-measurable
scalar valued functions on \( \Omega \) with the norm \( ||f||_p = (\int_\Omega |f|^p \, d\mu)^{1/p} \).
For a definition of "(unconditional) basic sequence", "equivalence of basic
sequences" etc., see e.g. [6]. Let us recall some operator ideals (see [8] and
[11]):

Denote by \( F(X) [F_+(X), F_-(X)] \) the operator ideals of all Fredholm
\([\phi_+ (X), \phi_- (X)]\) perturbations, which means that \( T \in F(X) [T \in F_+(X),
T \in F_-(S)] \) iff \( T + S \in \phi(X) [\phi_-(X), \phi_+(X)] \) for all \( S \in \phi(X) [S \in
\phi_+(X), S \in \phi_-(X)] \). There is some operator ideal \( F_0 \) in the sense of Pietsch,
such that \( F_0(X) = F(X) \) and that is (see [8, 5.3]).

\( F_0(X, Y) = \{ S : X \to X | Id_{X} + TS \in \phi(X) \text{ for all } T : Y \to X \} \). These ideals
will be compared with the following classes of operators [3].

Definition. Let \( \mathcal{F} \) be a class of infinite dimensional Banach spaces.

(a) An operator \( T : X \to Y \) is said to be \( \mathcal{F}\)-singular if there is no commuting
diagram

\[
\begin{array}{c}
X \\
\downarrow T \\
F \\
\uparrow J_1 \\
Y \\
\end{array}
\]

where \( F \in \mathcal{F} \) and \( J_1, J_2 \) are isomorphisms.

(b) \( T : X \to Y \) is said to be \( \mathcal{F}\)-cosingular if there is no commuting diagram

\[
\begin{array}{c}
X \\
\downarrow T \\
F \\
\uparrow Q_1 \\
Y \\
\end{array}
\]

where \( F \in \mathcal{F} \) and \( Q_1, Q_2 \) are surjective operators.

If \( \mathcal{F} \) contains all infinite dimensional Banach spaces, then the \( \mathcal{F}\)-singular
operators are called strictly singular (or semicompact [11, p. 252]) and the
\( \mathcal{F}\)-cosingular operators are called strictly cosingular (or cosemicompact [11, p.
257]). Denote by \( S [C] \) the ideals of strictly [co]singular operators. It is
remarked in [3] that the product of two operators, one of them being
(\mathcal{F}-(co)singular, is \mathcal{F}-(co)singular too (for arbitrary \mathcal{F}). But in general the class of \mathcal{F}-(co)singular operators is not an operator ideal (see [3]).

**Proposition.** (a) Let each infinite dimensional closed subspace of every \( F \in \mathcal{F} \) contain a subspace isomorphic to some element of \( \mathcal{F} \). Then the \( \mathcal{F} \)-(singular) operators form an ideal.

(b) Let each infinite dimensional quotient space of every \( F \in \mathcal{F} \) have a quotient space isomorphic to some element of \( \mathcal{F} \). Then the \( \mathcal{F} \)-(cosingular operators form an ideal.

**Proof.** By the preceding remark we only have to show that the sum of two \( \mathcal{F} \)-singular (\( \mathcal{F} \)-cosingular) operators is again \( \mathcal{F} \)-singular (\( \mathcal{F} \)-cosingular).

(a) Let \( S, T : X \to Y \) be \( \mathcal{F} \)-singular and let \( F \) be a subspace of \( X \) isomorphic to some element of \( \mathcal{F} \). Then \( S|_F \) and \( T|_F \) are strictly singular. Otherwise there exists an infinite dimensional subspace \( F_1 \) of \( F \), such that \( S|_{F_1} \) or \( T|_{F_1} \) is an isomorphism. But this contradicts the \( \mathcal{F} \)-singularity of \( S, T \) since \( F_1 \) contains a subspace isomorphic to some element of \( \mathcal{F} \). Hence, by [11, p. 253], \((S + F)|_F\) is strictly singular and cannot be an isomorphism.

(b) Similarly, by using [11, C II, Corollary 6.4].

In particular, the sets of \( (l_1, l_2) \)-(co)singular operators [6, II.2.2] and the class of \( c_0 \)-cosingular operators considered in [3] form operator ideals (because each quotient space of \( c_0 \) has itself a quotient space isomorphic to \( c_0 \) (see [5, p. 78])).

**III. Proofs.**

**Proof of Theorem.** Since (c) \( \Leftrightarrow \) (d) \( \Leftrightarrow \) (e) follows from (a) \( \Leftrightarrow \) (b) \( \Leftrightarrow \) (e) by duality, and trivially (a) \( \Rightarrow \) (b) \( \Rightarrow \) (e), it is enough to show (e) \( \Rightarrow \) (a). First we show that there is no loss of generality to assume that \( T : L_p(0, 1) \to L_p(0, 1) \).

Let \( M \) be a separable subspace of \( L_p(\Omega, \Sigma, \mu) \) such that \( T|_M \) is an isomorphism. By [14, Lemma 1] there exists a set \( \hat{\Omega} \in \Sigma \), a \( \sigma \)-subring \( \hat{\Sigma} \) of \( \Sigma \) restricted to \( \hat{\Omega} \) such that:

(i) \( L_p(\hat{\Omega}, \hat{\Sigma}, \mu) \) is separable, complemented in \( L_p(\Omega, \Sigma, \mu) \) and \( (\hat{\Omega}, \hat{\Sigma}, \mu) \) is \( \sigma \)-finite.

(ii) \( M \subset L_p(\hat{\Omega}, \hat{\Sigma}, \mu) \).

(iii) \( L_p(\hat{\Omega}, \hat{\Sigma}, \mu) \) is an invariant subspace for \( T \).

Use [13, §15.2] to find a space \( L \supset L_p(\hat{\Omega}, \hat{\Sigma}, \mu) \), \( L \) isomorphic to \( L_p(0, 1) \), and a projection \( P : L \to L_p(\hat{\Omega}, \hat{\Sigma}, \mu) \). Now if \( T_1 = TP : L \to L \) contradicts (e), then \( T : L_p(\Omega, \mu) \to L_p(\Omega, \mu) \) contradicts (e).

Hence it is enough to consider operators \( T : L_p(0, 1) \to L_p(0, 1) \), \( 1 < p < 2 \).

(The case \( 2 < p < \infty \) follows immediately from a theorem of Kadec-Pelczynski [6, II 3.10, 3.11]: An infinite dimensional subspace \( M \) of \( L_p(0, 1) \), \( 2 < p < \infty \), is either isomorphic to \( l_2 \) and complemented in \( L_p(0, 1) \), or it
contains a subspace which is isomorphic to $l_p$ and complemented in $L_p(0, 1)$.

Let $M$ be an infinite dimensional subspace of $L_p(0, 1)$ such that $T|_M$ is an isomorphism.

If $M$ contains some subspace isomorphic to $l_p$, then, by a theorem of Enflo-Rosenthal, [1, Theorem 3.1], $T(M)$ contains even a complemented (in $L_p(0, 1)$) subspace isomorphic to $l_p$. But this contradicts (e).

Now we assume that $M$ contains no subspace isomorphic $l_p$. By [6, I 1.14] there is an unconditional basic sequences $(f_n) \subset M$ with $\|f_n\| = 1$, $\|Tf_n\| > C > 0$.

We claim that for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that

\[(1) \quad \mu(A) < \delta(\epsilon) \Rightarrow \sup_n \int_A |f_n|^p \, d\mu < \epsilon.
\]

Otherwise there is a subsequence $(f_{n_k})$ of $(f_n)$, a sequence of measurable sets $(A_k)$ and $D > 0$ satisfying

\[\mu(A_k) < 1/2^k, \quad \int_{A_k} |f_{n_k}|^p \, d\mu > D, \quad k \in \mathbb{N}.
\]

Consider the sets $B_k = \bigcup_{j=K_k}^{\infty} A_j$. Since

\[\int_{B_k - B_{k+1}} |f_{n_k}|^p \, d\mu \rightarrow \int_{B_k} |f_{n_k}|^p \, d\mu > D
\]

there is a subsequence of indices $k_i$ such that

\[\int_{B_{k_i} - B_{k_{i+1}}} |f_{n_k}|^p \, d\mu > D.
\]

Since $(B_k - B_{k+1})_{i=\infty}$ are pairwise disjoint, it follows from [4, Lemma 2], that $(f_{n_k})$ is equivalent to the unit vector basis of $l_p$. This proves (1).

Next we construct a sequence $(g_n) \subset L_p(0, 1)$ such that

(2) $1/2 < \|g_n\| < 2$, $g_n \rightarrow^\ast 0$, $|g_n(t)| < M < \infty$ for all $t \in (0, 1)$, $\|T(g_n)\| > C/2$ and $(Tg_n)$ is an unconditional basic sequence.

(3) Since $(f_n)$ is bounded there is a $M_\epsilon > 0$ for every $\epsilon > 0$ such that

\[\mu(\{|f_n| > M_\epsilon\}) < \delta(\epsilon) \text{ for all integers } n.
\]

Define

\[\tilde{f}_n(t) = \begin{cases} f_n(t) & \text{provided } |f_n(t)| \leq M_\epsilon, \\ 0 & \text{otherwise.} \end{cases}
\]

By extracting a subsequence, if necessary, we may assume that $(f_n - \tilde{f}_n)$ converges weakly to some $f \in L_p(0, 1)$. Choose a measurable set $B$ such that

\[\mu(CB) < \delta(\epsilon), |f(t)| < N < \infty \text{ for } t \in B. \]

Because of (1), (3) we have $\|f_n - \tilde{f}_n\| < \epsilon$, $\|f\| < \epsilon$. Now defining $g_n = (\tilde{f}_n - f)\chi_B$, (2) is satisfied for $\epsilon$ small enough. (2) implies that some subsequence $(g_{n_k})$ of $(g_n)$ is an unconditional basic sequence in $L_2(0, 1)$. Therefore we have for all $\alpha_1, \ldots, \alpha_\epsilon \in \mathbb{R}$:
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\[ \left\| \sum_{i=1}^{r} \alpha_i T(g_k) \right\|_p < \| T \| \cdot \left\| \sum_{i=1}^{r} \alpha_i g_k \right\|_p \]
\[ < \| T \| \cdot \left( \sum_{i=1}^{r} |\alpha_i|^2 \right)^{1/2} \]

On the other hand, since \( T(g_k) \) is an unconditional basic sequence too, it follows from a theorem of Orlicz [6, II 3.7], that

\[ \left\| \sum_{i=1}^{r} \alpha_i g_k \right\|_2 < \left\| T \right\| \cdot C_1 \cdot \left( \sum_{i=1}^{r} |\alpha_i|^2 \right)^{1/2} \]

Finally, by [10, Theorem 2.1], \( \text{span}(T(g_k)) \) contains a subspace isomorphic to \( l_2 \) and complemented in \( L_p(0, 1) \). This again contradicts (e).

PROOF OF THE COROLLARIES. It is well known that \( S(X) \subset F_+(X) \subset F(X) \) and \( C(X) \subset F_-(X) \subset F(X) \) (see [2] and [11, Chapter V]). Assume that there exists some operator \( T \in F(X) - S(X) \) or \( T \in F(X) - C(X) \). By the theorem, there exists some subspace \( M \) of \( X \) and a projection \( P: X \to T(M) \) such that \( \ker X \to T \to \ker T(M) \) is an isomorphism onto \( T(M) \) and belongs to \( F(M, T(M)) \). This contradiction proves Corollary 1. Corollary 2 follows from the theorem and the simple fact that \( T \) has property (e) iff \( T' \) has property (e).

REMARK. The proof of Corollary 1 actually yields a slightly stronger result: For any proper operator ideal \( A \) in the sense of Pietsch (e.g. \( \text{Id} \notin A(Y) \) for all Banach spaces \( Y \)), we have: \( A(X) \subset S(X) \) for \( X = L_p(\mu) \).

ADDED IN PROOF. The referee kindly pointed out to me that the theorem is related to a result of H. P. Rosenthal (see Theorem 1, Convolution by a Biased Coin, The Altgeld Book 1975/76, University of Illinois, Functional Analysis Seminar):

Let \( T: L^1(0, 1) \to L^1(0, 1) \) be a bounded linear operator so that there is a sequence \( (f_n) \) in \( L^1(0, 1) \) which tends weakly to zero but \( (Tf_n) \) does not tend to zero in norm. Then there exists a subspace \( Y \) of \( L^1 \) isomorphic to \( l^2 \) with \( T|Y \) an isomorphism.

REFERENCES

3. W. Howard, \( \mathcal{F} \)-singular and \( \mathcal{F} \)-cosingular operators, Colloq. Math. 22 (1970), 85–89.

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