REMARKS ON A THEOREM OF KORANYI AND MALLIAVIN ON THE SIEGEL UPPER HALF PLANE OF RANK TWO

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ABSTRACT. In [5], A. Koranyi and P. Malliavin showed that bounded functions on the Siegel upper half plane of rank two which satisfied two special elliptic differential equations were characterized by their values on the Bergman-Silov boundary. In this paper a simple proof of this theorem is given.

1. Introduction. The Siegel upper half plane of rank 2, \( H_2 \), is defined as the space of \( 2 \times 2 \)-matrices \( Z \) with complex entries such that \( Z = Z' \) (' denotes transpose) and \( \text{Im} \ Z > 0 \) equipped with the metric

\[
\text{ds}^2 = \text{tr}\left((Z - Z^{-1})^{-1} dZ (Z - Z^{-1})^{-1} dZ\right).
\]

The Laplace operator associated to this metric is easily seen to be

\[
\Delta_1 = \text{tr}\left((Z - \bar{Z})^{-1} \partial_{\bar{Z}} (Z - \bar{Z}) \partial_Z\right)
\]

when for \( A = A' \), \( \partial_A = (\partial_y) \) with \( \partial_y = \frac{1}{2}(1 + \delta_y) \partial / \partial a_y \) and \( \partial_{\bar{Z}} \) does not differentiate the entries of \( Z - \bar{Z} \). Also, of interest to us is the operator

\[
\Delta_2 = \text{tr}\left(\partial_{\bar{Z}} (Z - \bar{Z}) \partial_Z\right).
\]

The Bergman-Silov boundary \( B_2 \) of \( H_2 \) contains the \( 2 \times 2 \) real symmetric matrices \( S_2 \) as an open dense subset and the Poisson kernel of \( H_2 \) with respect to \( B_2 \) is the function \( P_0: H_2 \times B_2 \rightarrow \mathbb{R}^+ \) such that

\[
P_0(Z, U) = c \left[ \frac{\text{det}(Z - \bar{Z})}{|\text{det}(Z - U)|^2} \right]^{3/2} \quad (Z \in H_2, U \in S_2).
\]

In [5], A. Koranyi and P. Malliavin showed that if \( f \in L^\infty(H_2) \) and \( \Delta_1 f = \Delta_2 f = 0 \)

\[
f(Z) = \int_{S_2} P(Z, U)F(U)dU
\]

for some \( F \in L^\infty(S_2) \). In this paper, we give a new proof of this result which uses the techniques found in [4] rather than diffusion processes. We first introduce the group of symplectic transformations.

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If $Sp(2, \mathbb{R})$ is the group of $4 \times 4$-matrices $X$ with real entries which have the property that $XJ = JX'$ for

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \quad (I_2 \text{ is } 2 \times 2 \text{ identity matrix}),$$

$Sp(2, \mathbb{R})$ acts transitively on $H_2$ as follows. For $Z \in H_2$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{R})$ ($A, B, C, D$ $2 \times 2$-matrices) then

$$g(Z) = (AZ + B)(CZ + D)^{-1}.$$  

Since $Sp(2, \mathbb{R})$ acts on $H_2$, it also acts on functions defined on $H_2$. If $f$: $H_2 \to \mathbb{C},$

$$Lgf(Z) = f(g^{-1}(Z)) \quad (g \in Sp(2, \mathbb{R})).$$

Then for $g \in Sp(2, \mathbb{R})$, $\Delta_1 Lgf = Lg\Delta_1 f$ and although $\Delta_2 Lgf \neq Lg\Delta_2 f$ in general it is true that if

$$g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \quad (X = X') \quad \text{or} \quad g = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad (A \in O(2))$$

$\Delta_2 Lgf = Lg\Delta_2 f$ and if

$$g = \begin{pmatrix} aI_2 & 0 \\ 0 & a^{-1}I_2 \end{pmatrix} \quad (a > 0)$$

$Lg\Delta_2 f = a^{-2}Lg\Delta_2 f$. Thus, if $K(H_2) = \{ f \in L^\infty(H_2): \Delta_1 f = \Delta_2 f = 0 \}$, $Lg$: $K(H_2) \to K(H_2)$ for

$$g = \begin{pmatrix} aA & aAX \\ 0 & a^{-1}A \end{pmatrix} \quad (X = X', A \in O(2), a > 0).$$

Here we point out that Koranyi and Malliavin [5] in fact consider the operator $\Delta_1 = c(tr(Z - \bar{Z}))\Delta_2$ and then $Lg\Delta_1 f = \Delta_1 Lgf$ for $g$ as above and $K(H_2) = \{ f \in L^\infty(H_2): \Delta_1 f = \Delta_2 f = 0 \}$.

If $f \in K(H_2)$ it follows from Furstenberg [1] that

$$f(Z) = \int_B P(Z, b)F(b)db$$

for some $F \in L^\infty(B)$ where $B$ is the Furstenberg boundary of $H_2$ and $P$: $H_2 \times B \to \mathbb{R}^+$ a Poisson kernel and $db$ is the $K = O(4) \cap Sp(2, \mathbb{R})$-invariant measure on $B$ normalized so that $\int_B db = 1$. After observing that $B = K/M$ where $M$ is the diagonal matrices in $K$, we have that $P$: $G \times K \to \mathbb{R}^+$ where

$$P(g, k) = c\| g^{-1}k\vec{e}_3 \|^{-2}\| g^{-1}k\vec{e}_3 \land \vec{e}_4 \|^{-2}.$$  

$\vec{e}_i$ is column vector with a 1 in the $i$th position 0's elsewhere and $\| \|$ is the norm on $\Lambda R^n$.

2. Reduction to a special case. Suppose now that $f \in K(H_2)$ and

$$f(Z) = \int_B P(Z, b)F(b)db \quad (F \in L^\infty(B)).$$
Let $K_0$ be the group of all matrices $g$ when
$$g = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad (A \in O(2))$$
and define for $\alpha \in C^\infty(K_0)$
$$\alpha \star_{K_0} f(Z) = \int_{K_0} \alpha(k)f(k^{-1}(Z))dk.$$ 
Then
$$\alpha \star_{K_0} f(Z) = \int_B P(Z, b)(\alpha \star_{K_0} F)(b)db.$$ 

Let $\hat{K}_0$ denote the set of equivalence classes of finite dimensional irreducible representations of $K_0$ and identify $\tau \in \hat{K}_0$ with its members. Then if we set $\chi_\tau(k) = (\deg \tau)\text{tr} \tau(k)$ we have
1. $\sum_{\tau \in \hat{K}_0} \chi_\tau \star_{K_0} f$ converges to $f$ in $C^\infty(H_2)$;
2. $\sum_{\tau \in \hat{K}_0} \chi_\tau \star_{K_0} F$ converges to $F$ in $L^2(B)$.
Since $B_2 = K/K_0$ to prove our result it suffices to show that each $\chi_\tau \star_{K_0} F$ is a function on $B_2$ or $S_2$.

Again let $\hat{N}$ be the group of all matrices $g$ where
$$g = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \quad (X = X').$$
Select $\{\phi_n; n \geq 0\} \subset C^\infty(\hat{N})$ so that $\phi_n \to \delta$. Then $\phi_n \star_\hat{N} f \to f$ as $n \to \infty$ in $C^\infty(H_2)$ and we see that
$$(\phi_n \star_\hat{N} f)(Z) = \int_B P(Z, b)(\phi_n \star_\hat{N} F)(b)db.$$ 
By observing that $\hat{N} \approx S_2$ and selecting our $\phi_n$ so that $\phi_n(AXA^{-1}) = \phi_n(X)$ for $A \in O(2)$, $X \in S_2$ we have that
$$\phi_n \star_\hat{N} \chi_\tau \star_{K_0} f = \chi_\tau \star_{K_0} \phi_n \star_\hat{N} f.$$ 
Writing $B = G/P$ for $P$ some subgroup of $G$ we have that $\hat{N}K_0P$ is open and dense in $B$ and $B \sim \hat{N}K_0P$ has measure 0. Thus almost everywhere we may express $F$ as an $L^\infty$ function on $S_2 \times K_0 \simeq S_2 \times S^1$
$$F(X; \theta) \quad (X \times e^{i\theta} \in S_2 \times S^1).$$
Convolving with a $\phi_n$ we have that if $D$ is any differential operator with constant coefficients in the $X$-variables sup$_{X, \theta}|DF(X, \theta)| < \infty$. After writing
$$X = \begin{pmatrix} x_1 + x_2 & x_3 \\ x_3 & x_1 - x_2 \end{pmatrix} \quad (x_1, x_2, x_3 \text{ real})$$
and setting $(x_2, x_3) = (r, 2\phi)$ (polar coordinates) we have that $F$ is a function in the variables $(x_1, r, \phi, \theta)$. Convolving $F$ over $K_0$ with some $\chi_\tau$ we have that
$$F(x_1, r, 2\phi + 2\psi; \theta + \psi) = e^{2i\phi}F(x_1, r, \phi; \theta)$$
for some integer $l$ and hence
$$F(x_1, r, 2\phi; \theta) = e^{2il\phi}F(x_1, r, 0; \theta - \phi).$$
Our result will follow if we can show that $F$ is independent of $\theta$.

3. **Main result.** For $Z \in H_2$ it will be convenient to use the coordinates

$$Z = \begin{pmatrix} z_1 + z_2 & z_3 \\ z_3 & z_1 - z_2 \end{pmatrix}$$

as was done in [5]. Then

$$\Delta_2^0 = c \left( y_1^2 \sum_{i=1}^{3} \tilde{\theta}_i \theta_i + \sum_{i=2}^{3} y_1 y_i \left( \tilde{\theta}_i \theta_1 + \tilde{\theta}_1 \theta_i \right) \right) = 0 \Delta_2 + \Delta_2^0$$

where

$$0 \Delta_2 = \frac{c}{4} \left( y_1^2 \sum_{i=1}^{3} \frac{\partial}{\partial x_i} + \sum_{i=2}^{3} 2y_1 y_i \frac{\partial}{\partial x_i} \right)$$

and

$$\Delta_2^0 = \frac{c}{4} \left( y_1^2 \sum_{i=1}^{3} \frac{\partial}{\partial y_i^2} + \sum_{i=2}^{3} 2y_1 y_i \frac{\partial}{\partial y_i} \right).$$

Suppose now that $f \in K(H_2)$ and

$$f(Z) = \int_{S^2 \times S^1} P(Z, X; \theta) F(X; \theta) dxd\theta$$

where $F(x_1; r, 2\phi; \theta) = e^{i4\phi} F(x_1; r, 0; \theta - \phi)$ and $\sup_{x, \theta} |DF(X; \theta)| < \infty$ for $D$ any differential operator in the $x$-variables with constant coefficients.

Let

$$H = \begin{bmatrix} -a_1 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix} \quad (a_1 > a_2 > 0)$$

and set $f_H(g) = \lim_{r \to \infty} f(g \text{ expt } H)$. Then we have the following lemma.

**Lemma 3.1.** $f_H = F$ almost everywhere and $(0 \Delta_2 f)_H = 0$ on $S^2 \times S^1$.

**Proof.** That $f_H = F$ almost everywhere is proved in Helgason and Koranyi [3]. That $(0 \Delta_2 f)_H = 0$ follows first by differentiating under the integral to obtain

$$\frac{\partial^2}{\partial x_i \partial x_j} f(Z) = \int_{S^1} \int_{S^2} \frac{\partial^2}{\partial x_i \partial x_j} P(Z; U; \theta) F(U; \theta) dU d\theta$$

and by the conditions on $F$ to obtain

$$\frac{\partial^2}{\partial x_i \partial x_j} f(Z) = \int_{S^1} \int_{S^2} P(Z; U; \theta) \frac{\partial^2}{\partial u_i \partial u_j} F(U; \theta) dU d\theta.$$
We are now in a position to obtain our main result. To do this we must examine the equation $(\Delta^2 f)_H = 0$. A few more preliminaries are necessary to accomplish this.

If $Z = X + iY \in H_2$, $Z = g_x g_T i$ for

$$g_x = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \quad \text{and} \quad g_y = \begin{pmatrix} Y^{1/2} & 0 \\ 0 & Y^{-1/2} \end{pmatrix}$$

but $Z = g_x g_T i$ for some

$$g_T = \begin{pmatrix} T & 0 \\ 0 & T^* \end{pmatrix} \quad \left( T = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \quad x, y, z \text{ real} \right)$$

An easy computation yields

$$y_1 = \frac{x^2 + (y^2 + 1)z^2}{2}, \quad y_2 = \frac{x^2 + (1 - y^2)x^2}{2}, \quad y_3 = yz^2,$$

$$\frac{\partial}{\partial y_1} = \frac{1 + y^2}{2x} \frac{\partial}{\partial x} - \frac{y}{x^2} \frac{\partial}{\partial y} + \frac{1}{2z} \frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial y_2} = \frac{1 - y^2}{2x} \frac{\partial}{\partial x} + \frac{y}{z^2} \frac{\partial}{\partial y} - \frac{1}{2z} \frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial y_3} = -\frac{y}{x} \frac{\partial}{\partial x} + \frac{1}{z^2} \frac{\partial}{\partial y}.$$

Modifying the arguments of Lemma 2.9 of [4] to see that $(\partial f/\partial x)_H = (\partial f/\partial z)_H = 0$ we obtain

**Lemma 3.2.** $(\Delta^2 f)_H = cd/dy((1 + y^2/2)^2 d/dy F(X; \tan^{-1} y)) = 0$.

Thus we obtain setting $\theta = \tan^{-1} y$ that $d/d\theta(\sec^2 \theta d/d\theta) F(X; \theta) = 0$ and solving we have

$$F(x_1; r, 2\phi; \theta) = F_1(x_1; r, 2\phi) + F_2(x_1; r, 2\phi; 0)(\sin 2\theta + 2\theta)$$

when $F_1$ is independent of $\theta$. Recall that

$$F(x_1; r, 2\phi; \theta) = e^{i\phi} F(x_1; r, 0; \theta - \phi).$$

Then

$$e^{i\phi} F(x_1; r, 0, \theta - \phi) = e^{i\phi} (F_1(x_1; r, 0) + F_2(x_1; r, 0; 0)(\sin 2(\theta - \phi) + (\theta - \phi))).$$

As $F(x_1; r, 2\phi; 0) = F_1(x_1; r, 2\phi)$ we have

$$F_1(x_1; r, 2\phi) = e^{i\phi} (F_1(x_1; r, 0) - F_2(x_1; r, 0; 0)(\sin 2\phi + 2\phi))$$

and so

$$e^{i\phi} (F_1(x_1; r, 0) + F_2(x_1; r, 0; 0)(\sin 2(\theta - \phi) + 2(\theta - \phi)))$$

$$= e^{i\phi} (F_1(x_1; r, 0) - F_2(x_1; r, 0; 0)(\sin 2\phi + 2\phi))$$

$$+ F_2(x_1; r, 2\phi; 0)(\sin 2\theta + 2\theta).$$
Thus
\[
F_2(x_1; r, 2\phi; 0)(\sin 2\theta + 2\theta)
= F_2(x_1, r, 0; 0)e^{i\phi} \left( (\sin 2\phi + 2\phi) + (\sin 2(\theta - \phi) + 2(\theta - \phi)) \right)
\]
and dividing by \(\sin 2\theta + 2\theta\) we see that the right-hand side of the equation is not independent of \(\theta\) unless \(F_2 = 0\) and finally we obtain the lemma.

**Lemma 3.3.** \(F\) is independent of \(\theta\).

**Remark.** That \(F_2 = 0\) also follows immediately from the boundedness of \(F\).
Finally we summarize.

**Theorem 3.4 (Koranyi and Malliavin [5]).** If \(f \in K(H_2)\)
\[
f(Z) = \int_{B_2} P_0(Z, b)F(b)db
\]
for some \(F \in L^\infty(B_2)\).

**References**


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