CONVERGENCE OF PROBABILITY MEASURES ON SEPARABLE BANACH SPACES

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ABSTRACT. The following result follows immediately from a general theorem on the convergence of probability measures on separable Banach spaces: On the space $C[0, 1]$ there exists a norm $p(x)$ equivalent to the ordinary norm such that if $\xi_1(t), \ldots, \xi_n(t), \ldots$ and $\xi(t)$ are continuous random processes $(0 < t < 1)$ and for any finite set of points $t_1, \ldots, t_k \subset [0, 1]$ the joint distribution of $p(\xi_1(t)), \xi_1(t_1), \ldots, \xi_n(t_k)$ converges to the joint distribution of $p(\xi(t)), \xi(t_1), \ldots, \xi(t_k)$, then $\xi_n(t)$ converges weakly to $\xi(t)$.

1. Let $X$ be a Banach space and $X^*$ be the dual space of $X$.

Definition [1, §2]. We say that $X$ has the $H$-property relative to a set $\mathcal{G} \subset X^*$ if from the conditions $\|x_n\| = \|x\| = 1$, $\lim_{n \to \infty} f(x_n) = f(x)$ for all $f \in \mathcal{G}$ it follows that $\lim_{n \to \infty} \|x_n - x\| = 0$.

In what follows we replace the phrase, “$X$ has the property $H$ relative to $\mathcal{G}$” by the symbol $X \in H_\mathcal{G}$.

Theorem 1 (Kadets, see [1, §2]). Let $X$ be a separable Banach space, and $\mathcal{G}$ be a subset of $X^*$ such that

$$\inf_{x \in X} \sup_{f \in \mathcal{G}} \frac{|f(x)|}{\|f\| \|x\|} = \eta(\mathcal{G}) > 0 \quad (x \neq 0, f \neq 0).$$

Then we can introduce in $X$ an equivalent norm $h(x)$ relative to which $X \in H_\mathcal{G}$.

It is easy to see that if $\eta(\mathcal{G}) > 0$, then there exists a countable set $\mathcal{G}_0 \subset \mathcal{G}$ such that $\eta(\mathcal{G}_0) > 0$.

In what follows in this section, we suppose that $X, \mathcal{G}, h(x)$ satisfy the conditions of Theorem 1 and $\mathcal{G}$ is a countable set.

Let

$$S = \{ x \in X : h(x) = 1 \}, \quad B = \{ x \in X : h(x) < 1 \}.$$

We denote by $L_\mathcal{G}(S)$ the Banach algebra$^1$ of bounded continuous functionals,$^2$ defined on $S$, which is generated by all functionals from $\mathcal{G}$, restricted to $S$.

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$^1$Throughout this paper, an algebra will mean an algebra with identity.

$^2$Throughout this paper, a functional defined on a set $A$ will mean a real function (without supplementary conditions) on $A$.

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We denote by $M_T(B)$ the Banach algebra of bounded continuous functionals, defined on $B$, which is generated by all functionals from $T$, restricted to $B$, and the functional $h(x)$, restricted to $B$.

**Lemma 1.** For any point $x_0 \in S$ and for any closed set $F \subset S$ ($x_0 \not\in F$) there exists $f \in L_T(S)$ such that $f(x_0) = 0$ and $f(x) \geq 1$ if $x \in F$.

**Proof.** Let $T = \{f_1, \ldots, f_k, \ldots\}$. Consider numbers $e_1 > e_2 > \ldots > e_k \ldots$, such that $e_k \to 0$. Let $U_k = \{x \in S: |f_i(x_0) - f_i(x)| < e_k, \ i = 1, \ldots, k\}$. There exists $k$ such that $U_k \cap F = \emptyset$. Define the functional $f$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{e_k^2} (f_i(x_0) - f_i(x))^2.$$  

Q.E.D.

**Lemma 2.** For any point $x_0 \in B$ and for any closed set $F \subset B$ ($x_0 \not\in F$) there exists $f \in M_T(B)$ such that $f(x_0) = 0$ and $f(x) \geq 1$ if $x \in F$.

**Proof.** Let $h(x_0) = a$. There exists a positive number $\varepsilon < a$ such that

$$U = \{x \in B: |h(x) - a| < \varepsilon, h(xa/h(x) - x_0) < \varepsilon\}$$

is contained in $B \setminus F$. Let

$$\hat{F} = \{x \in B \setminus U: h(x) = a\}.$$  

By Lemma 1 applied to $\hat{F}$ there exists $\hat{f} \in M_T(B)$ such that $\hat{f}(x_0) = 0$ and $\hat{f} > 1$ on $\hat{F}$. Since functionals in $M_T(B)$ are uniformly continuous, there exists a positive number $\delta < \min(\varepsilon, \frac{1}{2})$ such that $\hat{f}(x) > \frac{1}{2}$ if $|h(x) - a| < \delta$ and $xa/h(x) \not\in B \setminus U$. If $x \in F$, either $|h(x) - a| > \delta$, or $|h(x) - a| < \delta$ and $xa/h(x) \not\in B \setminus U$. Therefore we may take $f$ to be

$$f(x) = \frac{1}{\delta^2} \left[ (\hat{f}(x))^2 + (h(x) - a)^2 \right].$$  

Q.E.D.

In [2, §5] and in [3, §2] the following was proved:

**Lemma 3.** Let the probability measures $\mu_1, \ldots, \mu_n, \ldots$ and $\mu$ be defined on the Borel sets of the separable metric space $\Omega$ and let $G(\Omega)$ be a Banach algebra of bounded continuous functions defined on $\Omega$. Suppose that for any point $\omega_0 \in \Omega$ and for any closed set $F \subset \Omega$ ($\omega_0 \not\in F$) there exists $g \in G(\Omega)$ such that $g(\omega_0) = 0$ and $g(\omega) \geq 1$ for all $\omega \in F$. Suppose, finally, that for every function $g \in G(\Omega)$ the distribution of $g(\omega)$ with respect to the measure $\mu_n$ converges to the distribution of $g(\omega)$ with respect to the measure $\mu$. Then $\mu_n$ converges weakly to $\mu$.

Lemmas 2 and 3 imply the following:

**Theorem 2.** Let the probability measures $\mu_1, \ldots, \mu_n, \ldots$ and $\mu$ be defined on the Borel sets of $X$. If for any finite set of functionals $f_1, \ldots, f_k \in \Gamma$ the joint distribution of $h(x), f_1(x), \ldots, f_k(x)$ with respect to the measure $\mu_n$ converges to the joint distribution $h(x), f_1(x), \ldots, f_k(x)$ with respect to the measure $\mu$, then $\mu_n$ converges weakly to $\mu$. 

2. Consider the space $C[0, 1]$ with the ordinary norm $\|x(t)\| = \max_{0 < t < 1} |x(t)|$. Let $\Gamma$ be the set of functionals $f(x) = x(\tau)$, where $\tau$ is rational, $0 < \tau < 1$, $x \in C[0, 1]$. It is obvious that $\eta(\Gamma) = 1$. By virtue of Theorem 1 on the space $C[0, 1]$ there exists a norm $p(x)$, equivalent to the ordinary norm, relative to which $C[0, 1] \in H_\Gamma$. Theorem 2 implies the following:

**Theorem 3.** Let $\xi_1(t), \ldots, \xi_n(t), \ldots,$ and $\xi(t)$ be continuous random processes. If for any finite set of points $t_1, \ldots, t_k \subset [0, 1]$ the joint distribution of $p(\xi_n), \xi_n(t_1), \ldots, \xi_n(t_k)$ converges to the joint distribution of $p(\xi), \xi(t_1), \ldots, \xi(t_k)$, then for any continuous functional $f$ on $C[0, 1]$ the distribution of $f(\xi_n)$ converges to the distribution of $f(\xi)$.

One can give an explicit formula for the norm $p(x)$. For example

$$p(x) = \max_{0 < t < 1} |x(t)| + \sum_{n=1}^\infty \frac{1}{2^n} \max_{|r-s| < 1/n} |x(t) - x(s)|.$$

3. It is obvious that if we weaken the hypothesis of Theorem 3, and demand only that the joint distribution of $\xi_n(t_1), \ldots, \xi_n(t_k)$ converges to the joint distribution of $\xi(t_1), \ldots, \xi(t_k)$, then, in general, the conclusion no longer holds. Therefore, it is interesting to compare Theorem 3 to the following results:

**Theorem 4 [4].** Let $\xi_1(t), \ldots, \xi_n(t), \ldots,$ and $\xi(t)$ be measurable random processes ($0 < t < 1$). Suppose that there exist $C$ and $p > 1$ such that for all $n$ and $t$ we have $E[|\xi_n(t)|^p] < C$. Suppose, finally, that for any finite set of points $t_1, \ldots, t_k \subset [0, 1]$ the joint distribution of $\xi_n(t_1), \ldots, \xi_n(t_k)$ converges to the joint distribution of $\xi(t_1), \ldots, \xi(t_k)$ and $E[|\xi_n(t)|^p] \to E[|\xi(t)|^p]$ for all $t \in [0, 1]$. Then for any continuous functional $f$ on $L_p[0, 1]$ the distribution of $f(\xi_n)$ converges to the distribution of $f(\xi)$.

**References**


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