IDENTIFICATION OF CERTAIN 4-MANIFOLDS
WITH GROUP ACTIONS

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ABSTRACT. If $M^3$ is an orientable 3-manifold with an $S^1$-action or is a
Seifert fiber space, then the 4-manifold obtained by surgery along singular
fibers in $M \times S^1$ can also be obtained by surgery in $V^3 \times S^1$, where $V$ is a
manifold related to $M$ but with fewer singular fibers. An application is given
to Scharlemann’s “exotic” $(S^3 \times S^1 \# S^2 \times S^2)$’s.

Group actions have been successfully used in studying a wide class of 3
and 4-manifolds [6]–[9]. In this article we use circle actions to study the
geometric properties of a family of 4-manifolds obtained from $M^3 \times S^1$ by
performing surgery. Specifically, let $M^3$ be a local $S^1$-manifold without
$SE$-fibers [6], and in $M^3 \times S^1$ perform surgery along $k$ circles of the form
(exceptional orbit) $\times$ point. In Theorem 1 we show that the resulting mani-
fold can be obtained by surgery along $k$ principal orbits in $V^3 \times S^1$ where
$V^3$ is a local $S^1$-manifold with structure simpler than that of $M$. (It has $k$
fewer exceptional orbits.)

The class of 3-manifolds with local $S^1$-action includes all the Seifert
manifolds. If $M$ is a Seifert manifold with orbit space $S^2$ and $r$ exceptional
fibers and if $k > r - 2$, the above $V$ is a lens space. In particular, if $K$ is the
dodecahedral space, surgery on an exceptional orbit in $K \times S^1$ yields an
exotic 4-manifold studied by Scharlemann [10]. Theorem 1 implies that $W$
can be obtained by surgery along a principal orbit in some $S^1 \times \{\text{lens space}\}$.
We then use a theorem of Pao [8] to conclude that $W \# S^2 \times S^2$ is
diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2$.

1. Circle actions on 4-manifolds. In this section we review necessary pre-
liminary material concerning the classification of circle actions on closed
orientable 4-manifolds. Following [3] this is done in terms of a weighted
3-manifold which consists of the orbit space together with information about
the orbit types.

If $S^1$ acts locally smoothly on the closed oriented 4-manifold $W$ the orbit
space $W^*$ is an oriented 3-manifold. The orbits in $W$ which have nontrivial
finite cyclic isotropy groups are called exceptional, and their image $E^*$ in $W^*$

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consists of a finite collection of circles and open arcs. Each component of $E^*$ has constant orbit type and is oriented and assigned Seifert invariants $(\alpha, \beta)$. Thus if we identify a neighborhood of a point $x^* \in E^*$ with $I \times D^2$, $I \subseteq E^*$, then the $S^1$-submanifold of $W$ over $I \times D^2$ is equivalent to a product action $I \times (D^2 \times S^1)$ where $S^1$ acts on $D^2 \times S^1$ with Seifert invariants $(\alpha, \beta)$ [6].

If $F^*$ denotes the image in $W^*$ of the fixed point set then $E^* \cup F^*$ is the disjoint union of $\partial W^*$ and a collection of closed arcs, circles, and isolated points. To each closed arc in $E^* \cup F^*$ an ordered pair of the integers 0 and $-1$ is also assigned, but this can be ignored for the purposes of this paper.

There is one further invariant which is important for classifying the $S^1$-action, namely, the characteristic class. It is defined as follows. Let $Q^*$ be a regular neighborhood of the circles in $E^* \cup F^*$ and let $N^*$ be a regular neighborhood of the remainder $S^*$ of $E^* \cup F^*$. The restriction of the orbit map over $\text{Cl}(W^* - (Q^* \cup N^*))$ is a principal $S^1$-bundle projection which is trivial over $\partial Q^*$. This bundle can be extended over $Q^*$ by gluing in $Q^* \times S^1$ with a bundle isomorphism. There are infinitely many choices for the gluing bundle isomorphism, but a specific gluing map can be chosen in a natural way. (We refer the reader to [3, §9] for details.) Thus we determine a principal $S^1$-bundle over $\text{Cl}(W^* - N^*)$; let $\chi' \in H^2(\text{Cl}(W^* - N^*))$ be its Euler class. By Poincaré duality

$$H^2(\text{Cl}(W^* - N^*)) \cong H_1(\text{Cl}(W^* - N^*), \partial \text{Cl}(W^* - N^*))$$
$$\cong H_1(W^*, N^*) \cong H_1(W^*, S^*).$$

The Poincaré dual of $\chi'$, say $\chi \in H_1(W^*, S^*)$, is called the characteristic class of the $S^1$-action on $W$.

The weighted orbit space of the $S^1$-action on $W$ consists of the oriented orbit space $W^*$, the characteristic class $\chi \in H_1(W^*, S^*)$, and $E^* \cup F^*$ along with the orbit data described above. An isomorphism $f: W^*_1 \to W^*_2$ of weighted orbit spaces is an orientation-preserving homeomorphism which preserves orbit data and satisfies $f(\chi_1) = \chi_2$ where $\chi_1 \in H_1(W^*_1, S^*_1)$ is the characteristic class.

**EQUIVARIANT CLASSIFICATION THEOREM [3].** Oriented closed 4-manifolds with locally smooth $S^1$-actions are orientation-preserving equivariantly homeomorphic if and only if they have isomorphic weighted orbit spaces.

One sees from the constructions of [2] and [3] that each such action is equivalent to a smooth action. Also it follows easily from the proof of the classification theorem that if $W$ and $W'$ are smooth $S^1$-manifolds with isomorphic weighted orbit spaces there is a (not necessarily equivariant) PL homeomorphism $W \to W'$. Thus $W$ and $W'$ are diffeomorphic since $\text{PL} = \text{DIFF}$ in these low dimensions.

2. **Surgery on $M^3 \times S^1$**. One of the commonly used techniques of constructing interesting 4-manifolds consists of performing surgeries on a known 4-manifold along some properly chosen circles. The known 4-manifold
is frequently taken to be a bundle over $S^1$ with fiber a 3-manifold [1], [5], [9], [10]. In this section we study a family of 4-manifolds constructed in this way.

A local $S^1$-action on a space $X$ is a decomposition of $X$ into points and circles such that each decomposition element has a neighborhood admitting an effective $S^1$-action with the elements of the decomposition as orbits. A local $S^1$-action on a 3-manifold is characterized by its orbit invariants. We shall be concerned here with orbit invariants of the form \( \{ b; (e, g, h, 0); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r) \} \), which can be explained briefly as follows (see [6]). If $M$ is a local $S^1$-3-manifold with the above invariants, its orbit (decomposition) space is a 2-manifold $M^*$ of genus $g$ with $h$ boundary components, the images of $h$ circles of fixed points in $M$. $M^*$ is orientable if $e = o_1$ or $o_2$, nonorientable if $e = n_1, n_2, n_3$ or $n_4$. There are $r$ exceptional orbits in $M$ with Seifert invariants $(\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)$. The integer $b$ and $e$ classify up to weak equivalence an $S^1$-bundle over $\text{Int} M^*$ which is obtained by canonically replacing tubular neighborhoods of exceptional orbits with tubular neighborhoods of principal orbits. The local action is an action if and only if $e = o_1$ or $n_2$, and in this case the associated $S^1$-bundle is principal and $b$ is its Euler class.

Now let $M$ be a 3-manifold with the local $S^1$-action \( \{ b; (e, g, h, 0); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r) \} \), and let the 2-manifold $M^*$ be its orbit space. The 4-manifold $N = M \times S^1$ inherits a product local $S^1$-action, trivial in the $S^1$-factor. The orbit space of this local action is $N^* = M^* \times S^1$, and $E^*$ consists of $r$ oriented circles with Seifert invariants $(\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)$. Let $C \subseteq N$ be any circle orbit, and let $T$ be a tube of $C$; so $T \approx S^1 \times D^3$. Performing a surgery on $N$ along $C$ removes $T$ and fills in a copy of $D^2 \times S^2$. It is easily seen that this cutting and pasting can always be done equivariantly; i.e. we can define an $S^1$-action on $D^2 \times S^2$ so that the attaching map is $S^1$-equivariant (cf. [9, §3]). Therefore, after performing surgeries on $N$ along circle orbits, the resulting manifold again has a local $S^1$-action. On the orbit space level the surgery replaces the orbit space of $T$, a 3-cell, with the orbit space of $D^2 \times S^2$, a 3-cell or $S^2 \times I$. If $C$ is an exceptional orbit of type $(\alpha_i, \beta_i)$ the orbit space of $D^2 \times S^2$ will be a 3-cell. The effect of this surgery on the orbit space is to replace the circle in $E^*$ corresponding to $(\alpha_i, \beta_i)$ by the circle

![Diagram](https://example.com/diagram)

or the arc.
In the second case the surgery has replaced orbits corresponding to the missing arc by principal orbits having trivial isotropy.

**Theorem 1.** Let $M^3$ be the local $S^1$-manifold \( \{ b; (e, g, \tilde{h}, 0); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r) \} \) and let $W$ be a 4-manifold obtained from $M \times S^1$ by performing surgeries on the circles $C_i \times t_i$, $i = k + 1, \ldots, r$, where $C_i$ is the exceptional orbit in $M$ of type $(\alpha_i, \beta_i)$ and $t_i \in S^1$. Then there is a local $S^1$-manifold $V^3 = \{ a; (e, g, \tilde{h}, 0); (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k) \}$ such that $W$ is diffeomorphic to a 4-manifold obtained from $V \times S^1$ by doing surgeries along $r - k$ principal orbits.

**Proof.** Consider first the case where $M$ is an oriented $S^1$-manifold ($e = o_1$). It follows from the above discussion that $W$ is an $S^1$-manifold with orbit space $W^* = M^* \times S^1$, $E^* \cup F^*$ consists of $\partial W^*$, $k$ oriented circles in $E^*$ with Seifert invariants $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$, and $r - k$ circles and arcs in $E^* \cup F^*$ as described above. Viewing $M^* \times S^1$ as obtained from $M^* \times I$ by identifying $M^* \times 0$ with $M^* \times 1$, we picture $W^*$ in Figure 1, leaving unlabelled the $r - k$ circles and arcs of $E^* \cup F^*$.

![Figure 1](image1)

Applying the replacement trick of [9, §2] to these $r - k$ components of $E^* \cup F^*$ alters the $S^1$-action on $W$ so that the new orbit space $W^*_1$ consists of $W^*$ without the above $r - k$ circles and arcs and with $0 < j < r - k$ open 3 disks removed and $r - k - j$ pairs of fixed points in int $W^*_1$ (Figure 2).

![Figure 2](image2)
The manifold $W$ is determined up to diffeomorphism by this weighted orbit space and the characteristic class of the action. The dual characteristic class $\chi'$ lies in $H^2(W^* - S^*)$, which is embedded in the short exact sequence

$$0 \to H^2(M^* \times S^1) \to H^2(W^* - S^*) \to \ker \delta \to 0,$$

where $\delta$ is the Mayer-Vietoris coboundary arising from the sequence for $M^* \times S^1 = (W^* - S^*) \cup (\bigcup_i D^3)$, $t = j + 2(r - k - j)$. Since $\ker \delta \subseteq H^2(\bigcup_i S^2)$, the sequence splits and we identify $H^2(W^* - S^*) = H^2(M^* \times S^1) \oplus \ker \delta$ and $\chi' = (\xi, \eta)$.

We may choose circles $\{S_i\}$ in $M^*$ which generate $H^1(M^*)$ and which do not intersect $E^* \cup F^*$. The various applications of surgery and the replacement trick performed on $M \times S^1$ to obtain $W$ do not alter the trivial principal orbit bundle over each $S_i \times S^1$. Thus if $V^* = M^* \times 1$ in Figure 2, $\xi$ is determined by $\xi|V^*$ which in turn is determined by the integer $a$, where the $S^1$-submanifold $V$ of $W$ over $V^*$ has orbit invariants $\{a; (o_1, g, h, o); (a_1, \beta_1), \ldots, (a_k, \beta_k)\}$. The other component $\chi'$ is determined by the principal orbit bundles over the components of $\bigcup_i S^2$, all of which arise due to the replacement trick. Over each of the $j$ 2-spheres which are boundaries of collar neighborhoods of 2-spheres in $F^*$, the principal orbit bundle is trivial. The remaining $2(r - k - j)$ 2-spheres are the boundaries of regular neighborhoods of single point components of $F^*$. They are introduced in pairs by the replacement trick and the Euler numbers of the bundles over the associated 2-spheres occur in a $\pm$-pair. It then follows that the isomorphism type of the weighted orbit space $W^*_1$, and therefore the diffeomorphism type of $W$, depends only on the orbit data of $W^*_1$ and the $S^1$-action over $V^*$. For this reason we call $V^*$ a characteristic surface of $W^*_1$.

Let $C_1, \ldots, C_{k-1}$ be principal orbits in $V \times S^1$. By performing equivariant surgeries in $V \times S^1$ with properly chosen framings we can construct an $S^1$-manifold $W_2$ whose weighted orbit space $W^*_2$ is exactly $W^*_1$. In $V^* \times S^1$ there is a $V^* \times t$ which is not affected by the surgeries, and this gives rise to a characteristic surface $V^*_2 \subseteq W^*_2$. The action over $V^*_2$ is just that on $V$, hence $W$ and $W_2$ are diffeomorphic.

If $\epsilon = n_2$ a similar argument applies. Although the equivariant classification theorem of [3] does not generally cover $S^1$-actions on nonorientable 4-manifolds, the extension to the particular case at hand is trivial.

In case $\epsilon \neq o_1, n_2$ the replacement trick still gives another local $S^1$-action to $W$ with orbit space $W^*_1$ as in Figure 2. The equivariant classification theorem has an analogue in this particular situation. Using notation of §1, one obtains over $\text{Cl}(W^*_1 - N^*)$ a (nonorientable) $S^1$-bundle which is classified by its first Stiefel-Whitney class (viewed as a homomorphism $\omega$: $\pi_1(\text{Cl}(W^*_1 - N^*)) \to \mathbb{Z}_2$) and by its primary obstruction $o_2 \in H^2(\text{Cl}(W^*_1 - N^*); \mathbb{Z}')$ where coefficients are twisted by $\omega$. The pair of invariants $(\omega, o_2)$ replaces the characteristic class, and the classification then proceeds as in §1 except that unoriented Seifert invariants $(1 \leq \beta < \alpha/2)$ are required (cf. [11]). If a
characteristic surface is chosen as before, then \((\omega, \sigma_2)\) can be obtained from the corresponding invariants over \(V^*\); so the earlier proof applies to the general situation.

3. Scharlemann's manifolds. Let \(K\) be the dodecahedral space, and in \(K \times S^1\) let \(C \times \text{point}\) be an embedded circle representing a noncentral element of \(\pi_1(K)\). There are two framings for surgery along \(C \times \text{point}\) and we denote the resulting 4-manifolds by \(W_c\) and \(W'c\), distinguished by \(w_2(W_c) = 0\) and \(w_2(W'c) \neq 0\). It is known that for all such \(C\), \(W_c\) is diffeomorphic to \(S^3 \times S^1 \# CP^2 \# -CP^2\). There are 118 noncentral elements in \(\pi_1(K)\), and the corresponding manifolds \(W_c\) were introduced by M. Scharlemann [10] to construct fake homotopy structures on \(S^3 \times S^1 \# S^2 \times S^2\). We address the question of whether the manifolds \(W_c\) are diffeomorphic to \(S^3 \times S^1 \# S^2 \times S^2\) [4, Problem 4.15].

It is well known that \(K\) carries the \(S^1\)-action with orbit data \(\{-1; (o_1, 0, 0, 0); (2, 1), (3, 1), (5, 1)\}\). If \(C\) is an exceptional orbit in \(K\), it represents a noncentral element in \(\pi_1(K)\), and it follows from Theorem 1 that \(W_c\) can be obtained from \(L \times S^1\), for some lens space \(L = \{(a; (o_1, 0, 0, 0); (a_1, \beta_1), (a_2, \beta_2)\}\), by performing surgery along a principal orbit. In [8, III. 3 and III. 5] it is shown that surgery on \(L \times S^1\) along an exceptional orbit \(C'\) yields \(S^3 \times S^1 \# S^2 \times S^2\). After performing another surgery, on \(C\), which can now be viewed as lying in \(S^3 \times S^1 \# S^2 \times S^2\), we obtain \(S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2\). On the other hand, after doing surgery on \(C\), the orbit \(C'\) becomes homotopically trivial in \(W_c\); so by general position \(C'\) bounds an immersed disk with isolated double points which can be piped to the boundary along arcs and eliminated. Thus \(C'\) bounds a smooth 2-disk in \(W_c\), and surgery on \(C'\) yields \(W_c \# S^2 \times S^2\). Since \(C\) and \(C'\) are disjoint the order of the surgeries is irrelevant and we obtain

\[ W_c \# S^2 \times S^2 = S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2. \]

For arbitrary choices of circles \(C_1, C_2\) in \(K \times S^1\) representing noncentral elements of \(\pi_1(K)\), the above argument shows

\[ W_{c_1} \# S^2 \times S^2 = W_{c_2} \# S^2 \times S^2. \]

In conclusion:

**Theorem 2.** If \(C\) is a simple closed curve in \(K\) representing a noncentral element of \(\pi_1(K)\) then \(W_c \# S^2 \times S^2\) is diffeomorphic to \(S^3 \times S^1 \# 2(S^2 \times S^2)\).

Of course, we would like to show that \(W_c = S^3 \times S^1 \# S^2 \times S^2\), but explicit computations following the proof of Theorem 1 put this result slightly out of reach. For example, if \(C\) is the exceptional orbit in \(K\) with Seifert invariants \((5, 1)\) then \(W_c\) carries a circle action with orbit space equal to \(S^2 \times S^1\) minus an open 3-disk as in Figure 3.
The characteristic class of this action is $\chi = 0$. The manifold $S^3 \times S^1 \# S^2 \times S^2$ carries a circle action with exactly the same orbit space but with $\chi = -1$.

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