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IDENTIFICATION OF CERTAIN 4-MANIFOLDS WITH GROUP ACTIONS

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ABSTRACT. If M^3 is an orientable 3-manifold with an S^1-action or is a Seifert fiber space, then the 4-manifold obtained by surgery along singular fibers in $M \times S^1$ can also be obtained by surgery in $V^3 \times S^1$, where V is a manifold related to M but with fewer singular fibers. An application is given to Scharlemann's "exotic" $(S^3 \times S^1 \# S^2 \times S^2)$'s.

Group actions have been successfully used in studying a wide class of 3 and 4-manifolds **[6]**–**[9]**. In this article we use circle actions to study the geometric properties of a family of 4-manifolds obtained from $M^3 \times S^1$ by performing surgery. Specifically, let M^3 be a local S^1-manifold without SE-fibers **[6]**, and in $M^3 \times S^1$ perform surgery along k circles of the form (exceptional orbit) $\times$ point. In Theorem 1 we show that the resulting manifold can be obtained by surgery along k principal orbits in $V^3 \times S^1$ where V^3 is a local S^1-manifold with structure simpler than that of M. (It has k fewer exceptional orbits.)

The class of 3-manifolds with local S^1-action includes all the Seifert manifolds. If M is a Seifert manifold with orbit space S^2 and r exceptional fibers and if $k \geqslant r - 2$, the above V is a lens space. In particular, if K is the dodecahedral space, surgery on an exceptional orbit in $K \times S^1$ yields an exotic 4-manifold studied by Scharlemann **[10]**. Theorem 1 implies that W can be obtained by surgery along a principal orbit in some $S^1 \times$ (lens space). We then use a theorem of Pao **[8]** to conclude that $W \# S^2 \times S^2$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2$.

1. Circle actions on 4-manifolds. In this section we review necessary preliminary material concerning the classification of circle actions on closed orientable 4-manifolds. Following [3] this is done in terms of a weighted 3-manifold which consists of the orbit space together with information about the orbit types.

If S^1 acts locally smoothly on the closed oriented 4-manifold W the orbit space W^* is an oriented 3-manifold. The orbits in W which have nontrivial finite cyclic isotropy groups are called exceptional, and their image E^* in W^*

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consists of a finite collection of circles and open arcs. Each component of E^* has constant orbit type and is oriented and assigned Seifert invariants (α, β). Thus if we identify a neighborhood of a point $x^* \in E^*$ with $I \times D^2$, $I \subseteq E^*$, then the S^1-submanifold of W over $I \times D^2$ is equivalent to a product action $I \times (D^2 \times S^1)$ where S^1 acts on $D^2 \times S^1$ with Seifert invariants (α, β) [6].

If F^* denotes the image in W^* of the fixed point set then $E^* \cup F^*$ is the disjoint union of ∂W^* and a collection of closed arcs, circles, and isolated points. To each closed arc in $E^* \cup F^*$ an ordered pair of the integers 0 and -1 is also assigned, but this can be ignored for the purposes of this paper.

There is one further invariant which is important for classifying the S^1-action, namely, the characteristic class. It is defined as follows. Let Q^* be a regular neighborhood of the circles in $E^* \cup F^*$ and let N^* be a regular neighborhood of the remainder S^* of $E^* \cup F^*$. The restriction of the orbit map over $\mathrm{Cl}(W^* - (Q^* \cup N^*))$ is a principal S^1-bundle projection which is trivial over ∂Q^*. This bundle can be extended over Q^* by gluing in $Q^* \times S^1$ with a bundle isomorphism. There are infinitely many choices for the gluing bundle isomorphism, but a specific gluing map can be chosen in a natural way. (We refer the reader to [3, §9] for details.) Thus we determine a principal S^1-bundle over $\mathrm{Cl}(W^* - N^*)$; let $\chi' \in H^2(\mathrm{Cl}(W^* - N^*))$ be its Euler class. By Poincaré duality

$$\begin{aligned} H^2(\mathrm{Cl}(W^* - N^*)) &\approx H_1(\mathrm{Cl}(W^* - N^*), \partial \mathrm{Cl}(W^* - N^*)) \\ &\approx H_1(W^*, N^*) \approx H_1(W^*, S^*). \end{aligned}$$

The Poincaré dual of χ', say $\chi \in H_1(W^*, S^*)$, is called the characteristic class of the S^1-action on W.

The weighted orbit space of the S^1-action on W consists of the oriented orbit space W^*, the characteristic class $\chi \in H_1(W^*, S^*)$, and $E^* \cup F^*$ along with the orbit data described above. An isomorphism $f\colon W_1^* \to W_2^*$ of weighted orbit spaces is an orientation-preserving homeomorphism which preserves orbit data and satisfies $f(\chi_1) = \chi_2$ where $\chi_i \in H_1(W_i^*, S_i^*)$ is the characteristic class.

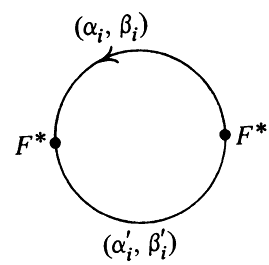
EQUIVARIANT CLASSIFICATION THEOREM [3]. *Oriented closed 4-manifolds with locally smooth S^1-actions are orientation-preserving equivariantly homeomorphic if and only if they have isomorphic weighted orbit spaces.*

One sees from the constructions of [2] and [3] that each such action is equivalent to a smooth action. Also it follows easily from the proof of the classification theorem that if W and W' are smooth S^1-manifolds with isomorphic weighted orbit spaces there is a (not necessarily equivariant) PL homeomorphism $W \to W'$. Thus W and W' are diffeomorphic since PL = DIFF in these low dimensions.

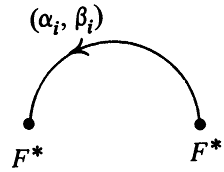
2. Surgery on $M^3 \times S^1$. One of the commonly used techniques of constructing interesting 4-manifolds consists of performing surgeries on a known 4-manifold along some properly chosen circles. The known 4-manifold is frequently taken to be a bundle over S^1 with fiber a 3-manifold **[1]**, **[5]**, **[9]**, **[10]**. In this section we study a family of 4-manifolds constructed in this way.

A local S^1-action on a space X is a decomposition of X into points and circles such that each decomposition element has a neighborhood admitting an effective S^1-action with the elements of the decomposition as orbits. A local S^1-action on a 3-manifold is characterized by its orbit invariants. We shall be concerned here with orbit invariants of the form $\{b; (\varepsilon, g, \bar{h}, 0); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}$, which can be explained briefly as follows (see **[6]**). If M is a local S^1-3-manifold with the above invariants, its orbit (decomposition) space is a 2-manifold M^* of genus g with $\bar{h}$ boundary components, the images of $\bar{h}$ circles of fixed points in M. M^* is orientable if $\varepsilon = o_1$ or o_2, nonorientable if $\varepsilon = n_1, n_2, n_3$ or n_4. There are r exceptional orbits in M with Seifert invariants $(\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)$. The integer b and ε classify up to weak equivalence an S^1-bundle over Int M^* which is obtained by canonically replacing tubular neighborhoods of exceptional orbits with tubular neighborhoods of principal orbits. The local action is an action if and only if $\varepsilon = o_1$ or n_2, and in this case the associated S^1-bundle is principal and b is its Euler class.

Now let M be a 3-manifold with the local S^1-action $\{b; (\varepsilon, g, \bar{h}, 0); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}$, and let the 2-manifold M^* be its orbit space. The 4-manifold $N = M \times S^1$ inherits a product local S^1-action, trivial in the S^1-factor. The orbit space of this local action is $N^* = M^* \times S^1$, and E^* consists of r oriented circles with Seifert invariants $(\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)$. Let $C \subseteq N$ be any circle orbit, and let T be a tube of C; so $T \cong S^1 \times D^3$. Performing a surgery on N along C removes T and fills in a copy of $D^2 \times S^2$. It is easily seen that this cutting and pasting can always be done equivariantly; i.e. we can define an S^1-action on $D^2 \times S^2$ so that the attaching map is S^1-equivariant (cf. [**9**, §3]). Therefore, after performing surgeries on N along circle orbits, the resulting manifold again has a local S^1-action. On the orbit space level the surgery replaces the orbit space of T, a 3-cell, with the orbit space of $D^2 \times S^2$, a 3-cell or $S^2 \times I$. If C is an exceptional orbit of type (α_i, β_i) the orbit space of $D^2 \times S^2$ will be a 3-cell. The effect of this surgery on the orbit space is to replace the circle in E^* corresponding to (α_i, β_i) by the circle



or the arc



In the second case the surgery has replaced orbits corresponding to the missing arc by principal orbits having trivial isotropy.

THEOREM 1. *Let M^3 be the local S^1-manifold $\{b;\ (\varepsilon, g, \bar{h}, 0);\ (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}$ and let W be a 4-manifold obtained from $M \times S^1$ by performing surgeries on the circles $C_i \times t_i$, $i = k + 1, \ldots, r$, where C_i is the exceptional orbit in M of type (α_i, β_i) and $t_i \in S^1$. Then there is a local S^1-manifold $V^3 = \{a;\ (\varepsilon, g, \bar{h}, 0);\ (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$ such that W is diffeomorphic to a 4-manifold obtained from $V \times S^1$ by doing surgeries along $r - k$ principal orbits.*

PROOF. Consider first the case where M is an oriented S^1-manifold ($\varepsilon = o_1$). It follows from the above discussion that W is an S^1-manifold with orbit space $W^* = M^* \times S^1$, $E^* \cup F^*$ consists of ∂W^*, k oriented circles in E^* with Seifert invariants $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$, and $r - k$ circles and arcs in $E^* \cup F^*$ as described above. Viewing $M^* \times S^1$ as obtained from $M^* \times I$ by identifying $M^* \times 0$ with $M^* \times 1$, we picture W^* in Figure 1, leaving unlabelled the $r - k$ circles and arcs of $E^* \cup F^*$.

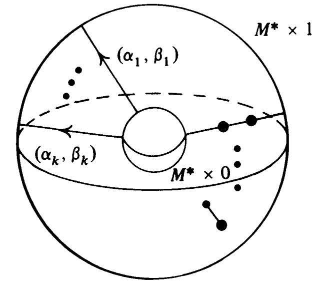


FIGURE 1

Applying the replacement trick of [9, §2] to these $r - k$ components of $E^* \cup F^*$ alters the S^1-action on W so that the new orbit space W_1^* consists of W^* without the above $r - k$ circles and arcs and with $0 \leqslant j \leqslant r - k$ open 3 disks removed and $r - k - j$ pairs of fixed points in int W_1^* (Figure 2).

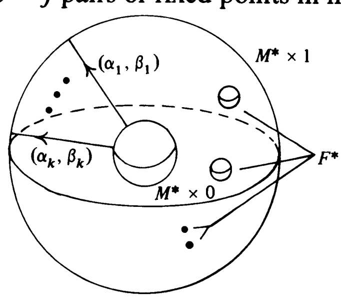


FIGURE 2

The manifold W is determined up to diffeomorphism by this weighted orbit space and the characteristic class of the action. The dual characteristic class χ' lies in $H^2(W^* - S^*)$, which is embedded in the short exact sequence

$$0 \to H^2(M^* \times S^1) \to H^2(W^* - S^*) \to \ker \delta \to 0,$$

where δ is the Mayer-Vietoris coboundary arising from the sequence for $M^* \times S^1 \cong (W^* - S^*) \cup (\dot{\bigcup}_t D^3)$, $t = j + 2(r - k - j)$. Since $\ker \delta \subseteq H^2(\dot{\bigcup}_t S^2)$, the sequence splits and we identify $H^2(W^* - S^*) = H^2(M^* \times S^1) \oplus \ker \delta$ and $\chi' = (\xi, \eta)$.

We may choose circles $\{S_i\}$ in M^* which generate $H_1(M^*)$ and which do not intersect $E^* \cup F^*$. The various applications of surgery and the replacement trick performed on $M \times S^1$ to obtain W do not alter the trivial principal orbit bundle over each $S_i \times S^1$. Thus if $V^* = M^* \times 1$ in Figure 2, ξ is determined by $\xi | V^*$ which in turn is determined by the integer a, where the S^1-submanifold V of W over V^* has orbit invariants $\{a; (o_1, g, \bar{h}, o); (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$. The other component η of χ' is determined by the principal orbit bundles over the components of $\dot{\bigcup}_t S^2$, all of which arise due to the replacement trick. Over each of the j 2-spheres which are boundaries of collar neighborhoods of 2-spheres in F^*, the principal orbit bundle is trivial. The remaining $2(r - k - j)$ 2-spheres are the boundaries of regular neighborhoods of single point components of F^*. They are introduced in pairs by the replacement trick and the Euler numbers of the bundles over the associated 2-spheres occur in a $\pm_1$-pair. It then follows that the isomorphism type of the weighted orbit space W_1^*, and therefore the diffeomorphism type of W, depends only on the orbit data of W_1^* and the S^1-action over V^*. For this reason we call V^* a characteristic surface of W_1^*.

Let $C_1', \ldots, C_{k-r}'$ be principal orbits in $V \times S^1$. By performing equivariant surgeries in $V \times S^1$ with properly chosen framings we can construct an S^1-manifold W_2 whose weighted orbit space W_2^* is exactly W_1^*. In $V^* \times S^1$ there is a $V^* \times t$ which is not affected by the surgeries, and this gives rise to a characteristic surface $V_2^* \subseteq W_2^*$. The action over V_2^* is just that on V, hence W and W_2 are diffeomorphic.

If $\varepsilon = n_2$ a similar argument applies. Although the equivariant classification theorem of [3] does not generally cover S^1-actions on nonorientable 4-manifolds, the extension to the particular case at hand is trivial.

In case $\varepsilon \neq o_1, n_2$ the replacement trick still gives another local S^1-action to W with orbit space W_1^* as in Figure 2. The equivariant classification theorem has an analogue in this particular situation. Using notation of §1, one obtains over $\mathrm{Cl}(W_1^* - N^*)$ a (nonorientable) S^1-bundle which is classified by its first Stiefel-Whitney class (viewed as a homomorphism ω: $\pi_1(W_1^*) = \pi_1(\mathrm{Cl}(W_1^* - N^*)) \to \mathbf{Z}_2$) and by its primary obstruction $o_2 \in H^2(\mathrm{Cl}(W_1^* - N^*); \mathbf{Z}')$ where coefficients are twisted by ω. The pair of invariants (ω, o_2) replaces the characteristic class, and the classification then proceeds as in §1 except that unoriented Seifert invariants ($1 \leqslant \beta \leqslant \alpha/2$) are required (cf. [11]). If a characteristic surface is chosen as before, then (ω, o_2) can be obtained from the corresponding invariants over V^*; so the earlier proof applies to the general situation.

3. Scharlemann's manifolds. Let K be the dodecahedral space, and in $K \times S^1$ let $C \times$ point be an embedded circle representing a noncentral element of $\pi_1(K)$. There are two framings for surgery along $C \times$ point and we denote the resulting 4-manifolds by W_c and W_c', distinguished by $w_2(W_c) = 0$ and $w_2(W_c') \neq 0$. It is known that for all such C, W_c' is diffeomorphic to $S^3 \times S^1 \# CP^2 \# - CP^2$. There are 118 noncentral elements in $\pi_1(K)$, and the corresponding manifolds W_c were introduced by M. Scharlemann **[10]** to construct fake homotopy structures on $S^3 \times S^1 \# S^2 \times S^2$. We address the question of whether the manifolds W_c are diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$ [**4**, Problem 4.15].

It is well known that K carries the S^1-action with orbit data $\{-1; (o_1, 0, 0, 0); (2, 1), (3, 1), ((5, 1))\}$. If C is an exceptional orbit in K, it represents a noncentral element in $\pi_1(K)$, and it follows from Theorem 1 that W_c can be obtained from $L \times S^1$, for some lens space $L = \{a; (o_1, 0, 0, 0); (\alpha_1, \beta_1), (\alpha_2, \beta_2)\}$, by performing surgery along a principal orbit. In [**8**, III. **3** and III. **5**] it is shown that surgery on $L \times S^1$ along an exceptional orbit C' yields $S^3 \times S^1 \# S^2 \times S^2$. After performing another surgery, on C, which can now be viewed as lying in $S^3 \times S^1 \# S^2 \times S^2$, we obtain $S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2$. On the other hand, after doing surgery on C, the orbit C' becomes homotopically trivial in W_c; so by general position C' bounds an immersed disk with isolated double points which can be piped to the boundary along arcs and eliminated. Thus C' bounds a smooth 2-disk in W_c, and surgery on C' yields $W_c \# S^2 \times S^2$. Since C and C' are disjoint the order of the surgeries is irrelevant and we obtain

$$W_c \# S^2 \times S^2 = S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2.$$

For arbitrary choices of circles C_1, C_2 in $K \times S^1$ representing noncentral elements of $\pi_1(K)$, the above argument shows

$$W_{c_1} \# S^2 \times S^2 = W_{c_2} \# S^2 \times S^2.$$

In conclusion:

THEOREM 2. *If C is a simple closed curve in K representing a noncentral element of $\pi_1(K)$ then $W_c \# S^2 \times S^2$ is diffeomorphic to $S^3 \times S^1 \# 2(S^2 \times S^2)$.*

Of course, we would like to show that $W_c = S^3 \times S^1 \# S^2 \times S^2$, but explicit computations following the proof of Theorem 1 put this result slightly out of reach. For example, if C is the exceptional orbit in K with Seifert invariants (5, 1) then W_c carries a circle action with orbit space equal to $S^2 \times S^1$ minus an open 3-disk as in Figure 3.

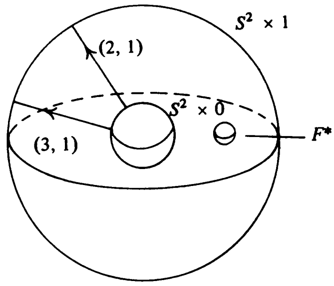


FIGURE 3

The characteristic class of this action is $\chi = 0$. The manifold $S^3 \times S^1 \# S^2 \times S^2$ carries a circle action with exactly the same orbit space but with $\chi = -1$.

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