THE MODULAR GROUP-RING OF A FINITE $p$-GROUP

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Abstract. For a finite $p$-group $G$ and the field $k$ of $p$ elements, we investigate the embedding of $G$ in the group $G^*$ of elements of the group-ring $kG$ having coefficient-sum equal to 1. Of particular interest is the question of when $G$ has a normal complement in $G^*$, for in this case simple proofs can be given for a number of diverse known results.

Since its inception in [5], the study of modular group-rings of finite $p$-groups has largely centred around the problem (see [1], [3], for example): “When does isomorphism of group-rings imply isomorphism of groups?” A key role in these investigations has been played by the group of units in the group-ring.

Fixing our notation, we let $p$ denote a fixed prime, $G$ a finite $p$-group, $k$ the field of $p$ elements and $kG$ the group-ring of $G$ over $k$. The group of units of $kG$ is simply $kG \setminus U$, where $U$ is the augmentation ideal of $kG$. Furthermore, $kG \setminus U = G^* \times k^*$, where $G^* = 1 + U$, and $k^* = k \setminus \{0\}$. We call $G^*$ the mod $p$ envelope of $G$, and denote the embedding $G \hookrightarrow G^*$ by $i_G$, or simply $i$. Note that $G^*$ is also a $p$-group.

Some properties of $i$ are as follows:

1. $Z(G) = Z(G^*) \cap G$;
2. $N_{G^*}(G) = GC_{G^*}(G)$;
3. $G' = (G^*)' \cap G$;
4. $\Phi(G) = \Phi(G^*) \cap G$.

None of these is very hard; (2) and (3) are proved in [2], and (4) in [6]. We have not yet, however, found a proof for

$$G^p = (G^*)^p \cap G.$$ 

Incidently, the truth of (2) was established independently by the present author in an (entirely abortive) attempt to find a simple proof of Gaschütz’ theorem [4].

We now define the class $\mathcal{C}_p$ of finite $p$-groups to consist of those $G$ which have a normal complement in $G^*$. The properties (1)–(5), along with many others, are immediately obvious for groups in $\mathcal{C}_p$, and it only remains to

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establish the extent of this class. The purpose of this article is to outline the present state of our knowledge about this problem.

The embedding $i: G \rightarrow G^*$ gives rise to a number of other questions which are beyond the scope of this article. For instance, what (if anything) can be said about the direct limit of the sequence $G \subseteq G^* \subseteq G^{**} \subseteq \cdots$? Can the embedding $G \subseteq G^*$ be defined by any useful universal property? Are there any significant differences in the theory when $k$ is replaced by an arbitrary field of characteristic $p$? Can we find a formula relating $\exp G$ and $\exp G^*$? While these are equal for abelian groups, equality does not hold in general, as the example of the elementary nonabelian group of order $p^3$ shows (see [6]).

It would also be of interest to investigate the relations between the modular representation theory (and also the cohomology) of $G$ and $G^*$, for if $E(G)$ denotes the representation algebra of $G$ (the vector space over the complex numbers with isomorphism classes of indecomposable $kG$-modules as a basis, and multiplication the Kronecker product of representations), then the definition of $G^*$ ensures that the restriction homomorphism: $E(G^*) \rightarrow E(G)$ splits.

Finally, and perhaps hardest of all, is there an algorithm for finding a presentation for $G^*$ (in terms of generators and relations) given a presentation of $G$?

**Theorem 1.** If $G$ is cyclic, then $G \in \mathbb{L}_p$.

**Proof.** There is an elementary result, which may be thought of as a lemma for the basis theorem or as a consequence of it, to the effect that for any finite abelian $p$-group $G$ and any $x \in G$ with $|x| = \exp G$, $\langle x \rangle$ is a direct factor of $G$. This, together with the obvious remark that $\exp G = \exp G^*$ when $G$ is abelian, proves the result.

We next give a simpler proof of a result in [6].

**Theorem 2.** If $G, H \in \mathbb{L}_p$, then $G \times H \in \mathbb{L}_p$.

**Proof.** The epimorphisms from $G \times H$ to $G$ and $H$ induce epimorphisms

$$
(G \times H)^* \xrightarrow{\nu_1} G^* \quad \xrightarrow{\nu_2} H^*
$$

If $N_1$ and $N_2$ are normal complements (which exist by hypothesis) for $G$ and $H$ in $G^*$ and $H^*$, respectively, then let $\bar{N}_1$ and $\bar{N}_2$ denote their pre-images in $(G \times H)^*$ under $\nu_1$ and $\nu_2$ respectively. $\bar{N}_1$ and $\bar{N}_2$ are clearly normal subgroups of $(G \times H)^*$ such that

$$
\bar{N}_1 \cap (G \times H) = H, \quad \bar{N}_2 \cap (G \times H) = G,
$$

so that if we let $N = \bar{N}_1 \cap \bar{N}_2$, then $N \cap (G \times H) = E$. Furthermore,
\[ |(G \times H)^\ast: N| < |(G \times H)^\ast: \overline{N_1}| < |(G \times H)^\ast: \overline{N_2}| = |G^\ast: N_1| \leq |H^\ast: N_2| = |G| \leq |G \times H|. \]

Thus \( N \) is the required normal complement.

**Theorem 3.** If \( G \) is abelian, then \( G \in \mathcal{L}_p \).

**Proof.** An immediate consequence of Theorems 1 and 2.

The next result is due to Tench [8], and yields the converse of Theorem 2.

**Theorem 4.** If \( G \) belongs to \( \mathcal{L}_p \), then so does any normally complemented subgroup \( H \) of \( G \).

**Proof.** Let \( \alpha: H^\ast \to G^\ast \) be the inclusion induced by \( H < G \), let \( \beta: G^\ast \to G \) be a splitting for \( \iota_G \), and let \( \gamma: G \to H \) be a splitting for \( H < G \). Then the composite

\[ H^\ast \xrightarrow{\alpha} G^\ast \xrightarrow{\beta} G \xrightarrow{\gamma} H \]

is clearly a splitting for \( \iota_H \), whose kernel is thus the required normal complement.

**Theorem 5.** For any \( G \), \( G^\ast \in \mathcal{L}_p \).

**Proof.** Note that \( G^\ast \), being a subset of \( kG \), is closed under the formation of linear combinations of its elements, provided the coefficient-sum is equal to 1. Now any element of \( G^{**} \) is just a “formal” linear combination of this type, and thus gives rise to a unique element of \( G^\ast \) (the corresponding “real” linear combination). This mapping is easily seen to be an epimorphism from \( G^{**} \) to \( G^\ast \) which fixes \( G^\ast \) elementwise. It is thus a splitting for \( \iota_{G^\ast} \) and its kernel is the required normal complement.

**Theorem 6.** If \( G_n \) denotes the Sylow \( p \)-subgroup of \( GL(n, p) \), then \( G_n \in \mathcal{L}_p \) for all \( n \).

**Proof.** The embedding \( G_n \to GL(n, p) \to M_n(k) \) extends by linearity to a homomorphism of rings \( kG_n \to M_n(k) \) whose restriction to \( G_n^\ast \) is such that all its images are units in \( M_n(k) \). We thus obtain a homomorphism \( \alpha: G_n^\ast \to GL(n, p) \) which fixes \( G_n \) elementwise. But since \( G_n^\ast \) is a \( p \)-group and \( G_n \) is a Sylow \( p \)-subgroup of \( GL(n, p) \), we must have \( \text{Im} \alpha = G_n \), so that \( \text{Ker} \alpha \) forms the required normal complement.

**Note.** By examining certain sets of upper triangular matrices with 1’s on the main diagonal, this argument can be extended to show that various other \( p \)-subgroups of \( GL(n, p) \) lie in \( \mathcal{L}_p \). Note further that if we knew \( \mathcal{L}_p \) to be subgroup-closed, it would follow from Theorem 6 (or from Theorem 5) that \( \mathcal{L}_p \) contained all finite \( p \)-groups.

Finally, for the sake of completeness, we list a few groups of small order in \( \mathcal{L}_p \) (see [7] for the proof).

**Theorem 7.** The following groups belong to \( \mathcal{L}_p \):

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(a) the two nonabelian groups of order 8 (p = 2),
(b) the nonabelian group of exponent p and order $p^3$ (p > 2),
(c) the three nonabelian indecomposable groups of exponent 4 and order 16 (p = 2).

Note in conclusion that the “smallest” group not definitely known to belong to $\mathcal{C}_p$ is the dihedral group $D_{16}$ of order 16 (p = 2). A programme involving the conjugacy classes of $D_{16}$ is currently in preparation to decide the question using a high-speed computing machine.

REFERENCES


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