PEAK SETS FOR LIPSCHITZ FUNCTIONS

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Abstract. We study the peak sets for the algebras of functions analytic in the unit disc $D$ and satisfying a Lipschitz condition on $\partial D$.

Let $D$ denote the open unit disc in the complex plane, let $D$ denote its closure, and let $\partial D$ denote its boundary. For $0 < \alpha < 1$, let $\text{Lip} \alpha$ be the algebra of complex-valued functions $f$ analytic on $D$, continuous on $\overline{D}$, and satisfying a Lipschitz condition of order $\alpha$ on $\partial D$:

$$|f(z) - f(w)| \leq K|z - w|^\alpha \quad (z, w \in \partial D).$$

Say that a function $f$ defined on $D$ peaks on $E \subseteq \partial D$ if $f = 1$ on $E$ and $|f| < 1$ on $D \setminus E$. Finally, say that $E \subseteq \partial D$ is a peak set for $\text{Lip} \alpha$ if some $f \in \text{Lip} \alpha$ peaks on $E$. We are interested in characterizing the peak sets for $\text{Lip} \alpha$. For $\alpha = 1$, the situation is easily described. It follows from our Theorem 1 that a peak set for $\text{Lip} 1$ must be finite. On the other hand, a result of B. A. Taylor and D. L. Williams [7] shows that any finite subset of $\partial D$ is a peak set for $\text{Lip} 1$. (In fact, [7] shows that the peaking function may be chosen to be infinitely differentiable on $\partial D$.) For $0 < \alpha < 1$, though, the situation seems more difficult, and we do not have such a characterization. Theorems 2 and 3 below give, respectively, sufficient and necessary conditions that $E \subseteq \partial D$ be a peak set for $\text{Lip} \alpha$ ($0 < \alpha < 1$). These conditions lend support to our conjecture of a necessary and sufficient condition, which we give at the end of this paper.

Before beginning, we establish some notation. We shall be dealing with closed subsets $E$ of $\partial D$, and we shall always assume, without loss of generality, that $-1 \in E$. For such an $E$, $\partial D \sim E$ is the union of a collection $\{(e^{a_n}, e^{b_n})\}$ of disjoint open arcs such that $-\pi < a_n < b_n < \pi$. We put $\epsilon_n = b_n - a_n$ and, when $E$ has been specified, shall use the $a_n$'s, $b_n$'s, and $\epsilon_n$'s without further comment. Now suppose that $f$ is a continuous function defined on $D$. We put $\|f\|_\infty$ equal to the (possibly infinite) number $\sup\{|f(z)|: z \in D\}$ and, for $0 < r < 1$, write

$$M(r, f) = \left(\frac{1}{2\pi}\right)\int_{-\pi}^\pi |f(re^{it})| \, dt.$$

**Theorem 1.** Suppose that $g$ is analytic in $D$, continuous on $\overline{D}$, and that
Re $g > 0$ in $D$. Let $N$ be the number of zeroes that $g$ has on $\partial D$. Then $N - 1 < 2\pi \| g' \|_\infty^2 / \| g(0) \|^2$.

Proof. Evidently we may assume that $\| g' \|_\infty < +\infty$ so that, in particular, $g' \in H^1$ and $g$ is absolutely continuous on $\partial D$ with $dg(e^{it})/dt = ie^{it}\lim_{r \to 1} g(re^{it})$ for almost every $t \in [-\pi, \pi]$. (See Theorem 3.11 in [2].)

Thus

$$(1) \quad |g(e^{it}) - g(e^{is})| < \| g' \|_\infty |t - s|, \quad -\pi < t, s < \pi.$$ 

Now let $E$ be the zero set of $g$ in $\partial D$. Given $n$, it follows from (1) and $g(e^{ia_n}) = g(e^{ib_n}) = 0$ that

$$(2) \quad |g(e^{it})| < \| g' \|_\infty \min \{t - a_n, b_n - t\} < \| g' \|_\infty \varepsilon_n / 2.$$ 

Let $u$ and $v$ be, respectively, the real and imaginary parts of $g$. Then the real part of $1/g$ is $u/|g|^2$. Thus

$$(3) \quad \frac{u(0)}{|g(0)|^2} = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} \frac{u(e^{it})}{g(e^{it})} dt = \left( \frac{1}{2\pi} \right) \sum \int_{a_n}^{b_n} \frac{u(e^{it})}{|g(e^{it})|^2} dt$$

$$\geq \left( \frac{1}{\pi} \right) \| g' \|_\infty^2 \sum \left( \frac{1}{\varepsilon_n} \right) \left[ \int_{a_n}^{b_n} \frac{u(e^{it})}{t - a_n} dt + \int_{(a_n + b_n)/2}^{b_n} \frac{u(e^{it})}{b_n - t} dt \right],$$

where the last inequality follows from (2).

It follows from (1) that the integral

$$\left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} u(e^{it}) \cot \left( \frac{a - t}{2} \right) dt$$

converges absolutely whenever $u(e^{ia}) = 0$. In fact, the formula for conjugate functions on the circle allows us to write, in this case,

$$v(e^{ia}) = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} u(e^{it}) \cot((a - t)/2) dt + v(0).$$

Since, in particular,

$$u(e^{ia_n}) = u(e^{ib_n}) = v(e^{ia}) = v(e^{ib}) = 0,$$

we have

$$\int_{-\pi}^{\pi} u(e^{it}) \left[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) \right] dt = 0.$$

Rewriting this we get

$$\int_{-\pi}^{a_n} + \int_{b_n}^{\pi} u(e^{it}) \left[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) \right] dt$$

$$(4) \quad = \int_{a_n}^{b_n} u(e^{it}) \left[ \cot \left( \frac{t - a_n}{2} \right) + \cot \left( \frac{b_n - t}{2} \right) \right] dt.$$

To examine the right-hand side of (4), we first note that for $a_n < t < (a_n +
\( b_n / 2 \) we have
\[
\left| \cot \left( \frac{b_n - t}{2} \right) \right| < \cot \left( \frac{t - a_n}{2} \right) < \frac{\pi}{t - a_n}.
\]

Thus
\[
\int_{a_n}^{(a_n + b_n) / 2} u(e^t) \left[ \cot \left( \frac{t - a_n}{2} \right) + \cot \left( \frac{b_n - t}{2} \right) \right] dt
\]
\[
< 2\pi \int_{a_n}^{(a_n + b_n) / 2} \frac{u(e^t)}{t - a_n} dt.
\]

Similarly,
\[
\int_{(a_n + b_n) / 2}^{b_n} u(e^t) \left[ \cot \left( \frac{t - a_n}{2} \right) + \cot \left( \frac{b_n - t}{2} \right) \right] dt
\]
\[
< 2\pi \int_{(a_n + b_n) / 2}^{b_n} \frac{u(e^t)}{b_n - t} dt.
\]

Now (3), (5), and (6) yield
\[
\frac{u(0)}{|g(0)|^2} > \frac{1}{2\pi^2 \| g' \|_{\infty}^2} \sum_n \frac{1}{\epsilon_n} \int_{a_n}^{b_n} u(e^t) \left[ \cot \left( \frac{t - a_n}{2} \right) + \cot \left( \frac{b_n - t}{2} \right) \right] dt.
\]

Taking into account (4), we have
\[
u(0) / |g(0)|^2
\]
\[
> \frac{1}{2\pi^2 \| g' \|_{\infty}^2} \sum_n \frac{1}{\epsilon_n} \int_{-\pi}^{a_n} + \int_{b_n}^{\pi} u(e^t) \left[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) \right] dt
\]
\[
= \frac{1}{2\pi^2 \| g' \|_{\infty}^2} \int_{-\pi}^{\pi} u(e^t) \left[ \sum_n \frac{1}{\epsilon_n} \left[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) \right] \cdot \chi_{[-\pi, a_n] \cup [b_n, \pi]}(t) \right] dt,
\]

where \( \chi_{[-\pi, a_n] \cup [b_n, \pi]} \) is the characteristic function of the set \([-\pi, a_n] \cup [b_n, \pi]\).

Now if \(-\pi < t < a_n \) or \(b_n < t < \pi\), it is easy to check that
\[
\cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) > \epsilon_n.
\]

Thus if \( t \in (a_m, b_m) \), the quantity \( \cdots \) in the last term of (7) is not less than \( \Sigma_{n \neq m} 1 / 2 \). Letting \( N \) be the cardinality (a priori possibly \( +\infty \)) of the collection \( \{ (e^{i\alpha_n}, e^{i\beta_n}) \} \), and noting that almost every \( t \in [-\pi, \pi] \) is in some \((a_m, b_m)\), we see from (7) that
\[
\frac{u(0)}{|g(0)|^2} > \frac{N - 1}{2\pi \|g\|_\infty^2} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) \, dt \right) = \frac{(N - 1)u(0)}{2\pi \|g\|_\infty^2}.
\]

This finishes the proof of the theorem.

**Corollary.** If \( E \) is a peak set for \( \text{Lip} \, 1 \), then \( E \) is finite.

**Proof.** Suppose that \( f \in \text{Lip} \, 1 \) peaks on \( E \). It follows from a result of Hardy and Littlewood (see Theorem 5.1 in [2]) that \( \|f'\|_\infty < \infty \), so Theorem 1 applies to \( g = 1 - f \).

Theorem 1 is a quantitative version of the following statement: if \( g \) is analytic in \( D \), continuous on \( \overline{D} \), and has positive real part on \( D \), and if \( \|g'\|_\infty \) is finite, then the zero set of \( g \) is finite. If the hypotheses on \( g \) are strengthened to require that \( g' \) be continuous on \( \overline{D} \), this statement is proved in [1]. It is perhaps surprising how much more difficult the proof becomes when \( f' \) is not required to be continuous on \( \overline{D} \).

The proof of our next theorem is similar to the proof of Theorem 5 in [1].

**Theorem 2.** Suppose \( 0 < \alpha < 1 \) and that \( E \subseteq \partial D \) is a closed set of measure zero satisfying \( \Sigma a_n^{(1 - \alpha)/(3 - \alpha)} < +\infty \). Then \( E \) is a peak set for \( \text{Lip} \, \alpha \).

**Proof.** Put \( \gamma = 2/(3 - \alpha) \) and define \( \phi \) on \( \partial D \) by

\[
\phi(e^{it}) = \begin{cases} 
(t - a_n)^{-\gamma} + (b_n - t)^{-\gamma} & \text{if } a_n < t < b_n, \\
+\infty & \text{if } e^{it} \in E.
\end{cases}
\]

Our hypothesis \( \Sigma a_n^{(1 - \alpha)/(3 - \alpha)} < +\infty \) implies that the function \( t \mapsto \phi(e^{it}) \) is integrable on \( [-\pi, \pi] \). Next define an analytic function \( g \) on \( D \) by

\[
g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(e^{it}) \frac{e^{it} + z}{e^{it} - z} \, dt.
\]

The properties of \( g \) that we need are these:

(i) \( g \) has a continuous extension to \( \overline{D} \sim E \),

(ii) \( \text{Re} \, g(re^{it}) \to \phi(e^{it}) \) as \( r \to 1 \),

(iii) \( |g(z)| \leq M[\text{dist}(z, E)]^{-2} \) for some constant \( M \),

(iv) \( g' \exp(-g) \) is bounded on \( D \).

For (i) and (ii) see [5, p. 80]; for (iii) see [8, Lemma 2.3]; and for (iv) see [6, pp. 1270–1271]. Now put \( h = 1/g \) and \( f = \exp(-h) \). Then \( h \) has a continuous extension to \( \overline{D} \), \( h \) has positive real part on \( \overline{D} \sim E \) (because \( g \) does), and \( h(e^{it}) = 0 \) for \( e^{it} \in E \). Thus \( f \) peaks on \( E \) and all that remains is to show that \( f \in \text{Lip} \alpha \). This will be done by showing that \( h' \) (and thus \( f' \)) belongs to the Hardy class \( H^p \), where \( p = 1/(1 - \alpha) \). (See, for example, exercise 9 on p. 91 in [2].)

Now \( h' = -h^2g' \exp(-g)/\exp(-g) \), so we see that \( h' \) is the quotient of two bounded analytic functions. Thus \( h' \) belongs to the Nevanlinna class \( N \). But because \( \exp(-g) \) is an outer function, \( h' \in N^+ \). Thus \( h' \in H^p \) provided the boundary function \( h'(e^{it}) \in L^p(\partial D) \). (See [2, Theorem 2.11].) Now
Thus it follows from (ii) and (iii) above that
\[
\limsup_{r \to 1} |h'(re^u)| < M[\text{dist}(e^u, E)]^{-2}[\phi(e^u)]^{-2}.
\]
There is a constant $K$ such that $[\text{dist}(e^u, E)]^{-\gamma} < K\phi(e^u)$, and so
\[
[\text{dist}(e^u, E)]^{-2} < [K\phi(e^u)]^{2/\gamma}.
\]
When combined with (8), this implies that
\[
|h'(e^u)| < MK^{2/\gamma}[\phi(e^u)]^{(2/\gamma)-2} \quad \text{a.e.}
\]
Consequently,
\[
|h'(e^u)|^p < \text{constant} \cdot \phi(e^u) \quad \text{a.e.}
\]
This shows that $h'(e^u) \in L^p(\partial D)$ and so completes the proof of the theorem.

To prove our final theorem we require a lemma.

**Lemma.** Let $g$ be analytic on $D$ and have positive real part. Then
\[
M(r, g) = O(\log [1/(1 - r)]) \quad \text{as } r \to 1.
\]
**Proof.** This is a consequence of [3, Theorem 7].

**Theorem 3.** Fix $\alpha$ with $0 < \alpha < 1$ and suppose $E \subseteq \partial D$ is a peak set for Lip $\alpha$. Then for each $\delta > 1$ we have
\[
\sum_n \epsilon_n^{1-\alpha}\log(1/\epsilon_n)\epsilon_n^{-\delta} < +\infty.
\]
**Proof.** Let $\delta > 1$ be given. Let us assume, without loss of generality, that $b_n - a_n < \pi/2$ for each $n$ so that if $r_0$ is the smallest of the numbers $\cos(b_n - a_n)$, then $0 < r_0 < 1$. Since $E$ is a peak set for Lip $\alpha$, there exists $f \in \text{Lip } \alpha$ having positive real part on $\partial D \sim E$ with $f(e^u) = 0$ for $e^u \in E$. Let $K$ be a Lipschitz constant for $f$ on $\overline{D}$ so that $|f(z) - f(w)| < K|z - w|^{\alpha}$ ($z, w \in \overline{D}$). (The assumption that $f \in \text{Lip } \alpha$ means that $f$ satisfies a Lipschitz condition on $\partial D$. But an old result of Hardy and Littlewood [4, Theorem 41] shows that $f$ is then Lipschitz on $\overline{D}$.)

By elementary calculus, $\int_0^1/(1 - r)[\log 1/(1 - r)]^{\delta} \, dr < +\infty$. When combined with the Lemma as applied to $g = 1/f$, this yields
\[
\int_{r_0}^1 M(r, 1/f)/(1 - r)[\log 1/(1 - r)]^{1+\delta} \, dr < +\infty.
\]
Thus,
\[
+ \infty > \int_{r_0}^{1} \left[ \int_{-\pi}^{\pi} \frac{1}{f(r e^{it})} \, dt \right] \, dr \\
= \int_{r_0}^{1} \left[ \sum_n \int_{a_n}^{b_n} \frac{1}{f(r e^{it}) - f(e^{ia_n})} \, dr \right] \, dr \\
= \sum_n \int_{a_n}^{b_n} \left[ \int_{r_0}^{1} \frac{dr}{f(r e^{it}) - f(e^{ia_n})} \, dr \right] \, dt \\
> \sum_n \int_{a_n}^{b_n} \left[ \int_{r_0}^{1} K|e^{re^{it}} - e^{ia_n}| (1 - r) \left[ \log 1 / (1 - r) \right]^{1+\delta} \, dr \right] \, dt.
\]

(The last inequality follows from the facts \(r_0 < \cos(b_n - a_n) < \cos(t - a_n)\) if \(a_n < t < b_n\), and \(|re^{it} - e^{ia_n}| < |e^{it} - e^{ia_n}|\) if \(\cos(t - a_n) < r < 1\). Evaluating \(\int_{\cos(t - a_n)}^{1} (1 - r) \left[ \log 1 / (1 - r) \right]^{1+\delta} \, dr\), we see that the last sum above is equal to

\[
\left(\frac{1}{K\delta}\right) \sum_n \int_{a_n}^{b_n} \frac{dt}{|e^{it} - e^{ia_n}| \left[ \log 1 / (1 - \cos(t - a_n)) \right]^{\delta}} \\
> \left(\frac{1}{2^\delta K\delta}\right) \sum_n \int_{a_n}^{b_n} \frac{dt}{(t - a_n) \left[ \log \pi / \sqrt{2} (t - a_n) \right]^{\delta}},
\]

where in obtaining the inequality we have used the relations

\[
|e^{it} - e^{ia_n}| < t - a_n \quad \text{and} \quad 1 - \cos(t - a_n) > 2(t - a_n)^2 / \pi^2.
\]

It follows, after a change of variable, that

\[
(9) \quad + \infty > \sum_n \int_{0}^{\sqrt{2} e_n / \pi} t^{-\alpha} \left[ \log 1 / t \right]^{-\delta} \, dt.
\]

Consider the equation

\[
(d/dt)(t^{1-\alpha} \left[ \log 1 / t \right]^{-\delta}) = (1 - \alpha)t^{-\alpha} \left( \log 1 / t \right)^{-\delta} + \delta t^{-\alpha} \left( \log 1 / t \right)^{-\delta-1}.
\]

By integrating both sides of this equation from 0 to \(\sqrt{2} e_n / \pi\) and then summing over \(n\), we obtain
Each sum on the right-hand side of this equation is finite because of (9). Thus so is the left-hand side. It follows easily that \( \Sigma_n e_n^{1-\alpha} < +\infty \). This completes the proof of the theorem.

In conclusion, we conjecture that the condition \( \Sigma_n e_n^{1-\alpha} < +\infty \) is necessary and sufficient for a closed subset \( E \) of \( \partial D \) having measure 0 to be a peak set for Lip \( \alpha \).

REFERENCES