NEAR COMPACTNESS AND SEPARABILITY
OF SYMMETRIZABLE SPACES

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Abstract. Although every feebly compact, Baire, semimetrizable space is separable, we prove here that for every infinite cardinal number \( n \) there exists a feebly compact, Baire, symmetrizable Hausdorff space which has no dense subset of cardinality less than \( n \).

For a topological space \( X \), a mapping \( d: X \times X \to [0, \infty) \) is said to be a symmetric provided that: (i) for all \( x, y \in X \), \( d(x, y) = d(y, x) \), and \( d(x, y) = 0 \) if and only if \( x = y \); and (ii) for any subset \( V \) of \( X \), \( V \) is open if and only if for each point \( v \in V \) there exists \( e > 0 \) with \( B(v, e) = \{ x \in X : d(x, v) < e \} \subset V \). If, in addition, each \( B(x, e), x \in X, e > 0 \), is a neighborhood of \( x \), then \( d \) is called a semimetric. A space \( X \) which has a symmetric (semimetric) is said to be symmetrizable (semimetrizable).

A. V. Arhangel'skii [A, p. 126] proved that every countably compact symmetrizable Hausdorff space is metrizable, and in [S1] and [S2] properties of symmetrizable feebly compact spaces were studied (recall that a space \( X \) is said to be feebly compact if every locally finite family of open subsets of \( X \) is finite). Of particular interest there was the question: Is every feebly compact symmetrizable space separable? Proofs were given in [S1] that every feebly compact symmetrizable space having a dense set of isolated points is separable, and in [S2, Theorem 10] that every feebly compact, Baire, semimetrizable space is separable. The latter extended Reed’s theorem [R] that every Moore-closed space is separable, for a Moore-closed space is regular and feebly compact [G], and a regular, feebly compact space is Baire [M].

In this paper, a modification of a very nice technique developed in [DGN, Example 3.1] is used to settle the question in the negative, and we obtain the following surprising result.

Theorem. Let \( n \) be an infinite cardinal number. Then there exists a Baire, feebly compact, symmetrizable Hausdorff space \( X \) such that no dense subset of \( X \) has cardinality less than \( m = n^\aleph_0 \).

Proof. Let \( Y \) be a metrizable Baire space such that \( |V| = m \) for every nonempty open subset \( V \) of \( Y \), and \( |D| = m \) for any dense subset \( D \) of \( Y \). Let \( d' \) be a metric for \( Y \), \( \mathfrak{B} \) a base for \( Y \) with \( |\mathfrak{B}| = m \), and \( C \) be the family of all
countably infinite, pairwise disjoint, locally finite families of nonempty members of $\mathcal{B}$.

List the members of the collection $\mathcal{C}$ as $\mathcal{C} = \{C_k: k < m\}$ and list the members of each $C_k$ in a 1-1 manner as $\bar{C}_k = \{C_{kj}: j \in \mathbb{N}\}$. Since each $|C_{kj}| = m$, one can by transfinite induction select points $s_{kj} \in C_{kj}$, where $k < m$ and $j \in \mathbb{N}$, so that whenever $i, k \in m$ and $i \neq k$, then

$$\{s_{kj}: j \in \mathbb{N}\} \cap \{s_{kj}: j \in \mathbb{N}\} = \emptyset.$$

Let $X = Y \cup m$ and extend $d'$ to a symmetric $d$ on $X$ by the rule

$$d(x, y) = d(y, x) = \begin{cases} 0 & \text{if } x = y; \\ d'(x, y) & \text{if } x, y \in X; \\ 1/j & \text{if } x = k \text{ and } y = s_{kj}; \text{ and} \\ 1 & \text{otherwise}. \end{cases}$$

Next, let $X$ have the topology induced on it by $d$.

Before verifying that $X$ is Hausdorff, observe that for each point $y \in Y$, one has $y = s_{kj}$ for at most one pair $k, j$, so for each $y \in Y$ there exists $e(y) > 0$ with $B(y, e(y)) \subset Y$. Thus $\{B(y, e): 0 < e < e(y)\}$ is a fundamental system of open neighborhoods of $y$ in $X$. For a point $k < m$, a fundamental system of open neighborhoods is the family of all sets having the form

$$\{k\} \cup \left( \bigcup \{B(s_{kj}, f_j): 0 < f_j < e(s_{kj}), t < j\} \right),$$

where $t \in \mathbb{N}$ and $f$ is a sequence of real numbers, the $j$th term of which is $f_j$.

Consider distinct points $x$ and $y$ in $X$. If both are in the metrizable open subset $Y$, then disjoint neighborhoods can certainly be found. Suppose $x = k < m$ and $y \in Y$. For some $t \in \mathbb{N}$, $\{y\}$ and $\{s_{kj}: j > t\}$ are disjoint closed subsets of $Y$ (since $\bar{C}_k$ is locally finite in $Y$ and pairwise disjoint), so there exist disjoint open subsets $U$ and $V$ of $Y$ with $y \in U$ and $\{s_{kj}: j > t\} \subset V$. Thus, $U$ and $V \cup \{k\}$ are disjoint neighborhoods of $y$ and $x$. If $x = k$ and $y = r$ with $k, r < m$, then one can (again) appeal to the normality of $Y$ and topology on $X$ to find disjoint open sets containing $\{k\} \cup \{s_{kj}: j \in \mathbb{N}\}$ and $\{r\} \cup \{s_{kj}: j \in \mathbb{N}\}$.

Because $Y$ is a dense, Baire subspace of $X$, the space $X$ must also be Baire. Since $Y$ is an open subspace having no dense subset of cardinality less than $m$, then $X$ has no dense subset of cardinality less than $m$.

Finally, suppose that $\mathcal{V}$ is an infinite family of open subsets of $X$. We will prove that $\mathcal{V}$ fails to be locally finite.

Suppose, on the contrary, that $\mathcal{V}$ is locally finite. Since $Y$ is dense in $X$, one can find a countably infinite pairwise disjoint family $\mathcal{W}$ of members of $\mathcal{B}$ and a 1-1 mapping $f: \mathcal{W} \rightarrow \mathcal{V}$ such that for each $W \in \mathcal{W}$, one has $W \subset f(W)$. Evidently any point at which $\mathcal{W}$ fails to be locally finite must also be a point at which $\mathcal{V}$ fails to be locally finite. Thus $\mathcal{W}$ is locally finite with respect to $Y$, and hence $\mathcal{W} = \bar{C}_k$ for some $k < m$. But clearly $\bar{C}_k$ fails to be
locally finite at the point $k$, so we have a contradiction.

**Remarks.** (i) I do not know if every regular, feebly compact symmetrizable space is separable. Since a $G_\delta$-point in a regular, feebly compact space must have a countable neighborhood base (by an observation of I. Glicksberg), and since a first countable symmetrizable Hausdorff space is semimetrizable, any example of a regular, feebly compact, symmetrizable space that is not separable would also provide a negative answer to the still open question (see [DGN]) as to whether or not every point of a regular symmetrizable space must be a $G_\delta$.

(ii) In the construction above, if $Y$ is chosen so that no compact subset of $Y$ has nonempty interior, then arguments similar to ones given in [DGN] show that $X$ has a closed subset, namely $m$, which fails to be a $G_\delta$-set (because then if $\mathcal{V}$ is a countable family of open sets containing $m$, the family $\mathcal{U} = \{ V \cap Y : V \in \mathcal{V} \}$ consists of dense open subsets of $Y$, and so $\emptyset \neq \bigcap \mathcal{U} \subset Y$ and $\bigcap \mathcal{V} \neq m$).

**Bibliography**


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