RESIDUAL LINEARITY FOR CERTAIN NILPOTENT GROUPS

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Abstract. In this note we consider relations between residual finiteness and residual linearity for a nilpotent group $G$. We show, amongst other things, that if the center and the commutator subgroup of $G$ are finitely generated and $G$ is residually linear, then $G$ is residually finite. Indeed the property which we use on linear groups is that linear groups satisfy the minimal condition on centralizers.

Let $P$ be a group property. We recall that a given group $G$ is called residually $P$ if it is a subdirect product of groups having the property $P$. For any group $G$ let $R(G)$ be the intersection of all its normal subgroups of finite index. Thus $R(G) = \langle 1 \rangle$ if and only if $G$ is residually finite. For any integer $n$ let $G^n$ be the subgroup of $G$ generated by the $n$th powers of elements of $G$. For abelian groups it is well known that $R(G) = \cap_{n>1} G^n$; if in addition the $p$-torsion of $G$ is bounded for each prime $p$, then $R(G)$ is a radicable group. We say that $G$ is residually linear if for each $1 \neq x \in G$ there exists a field $K$ and a homomorphism $\phi: G \to \text{GL}(n, K)$ such that $\phi(x) \neq 1$. An abelian group is $\mathbb{Z}$ if it is a subdirect product of cyclic groups $C_i$ such that $C_i = \mathbb{Z}$ or $|C_i| < n$ for a fixed integer $n$. We will use the symbols $\Gamma_n(G)$ and $Z_n(G)$ for the terms in the lower and upper central series of $G$. If $X$ is a subset of the group $G$ we denote by $C_G(X)$ its centralizer.

The main result of this paper is

Theorem I. (i) Let $G$ be a nilpotent residually linear group. If $\Gamma_2(G)$ is finitely generated and $Z_1(G)$ is $\mathbb{Z}$, then $G$ is residually finite.

(ii) There exists a nilpotent group of class 2 with $\Gamma_2(G)$ finitely generated and $Z_1(G)$ residually finite, such that it is residually linear but it is not residually finite.

(iii) There exists a nilpotent group of class 3 with $Z_1(G)$ cyclic, which is residually linear but it is not residually finite.

Corollary. Let $G$ be a nilpotent group of class 2 with $Z_1(G)$ finitely generated. Then residually linear implies residually finite.

Lemma I. Let $G$ be a nilpotent linear group. If $H$ is a normal subgroup of $G$ such that $H \cap Z_1(G)$ is finitely generated, then $H$ is finitely generated.

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Proof. We proceed by induction on the class $c$ of $G$, the case $c = 1$ being obvious. Suppose that the elements $x_1, x_2, \ldots, x_n$ of $G$ span $G$ linearly, clearly

$$Z_1(G) = C_G(x_1, x_2, \ldots, x_n).$$

Let $[x, y] = x^{-1}y^{-1}xy$ denote commutators in $G$, then the map

$$H \cap Z_2(G) \to (H \cap Z_1(G)) \times \cdots \times (H \cap Z_1(G))$$

in which $x \mapsto ([x, x_1], \ldots, [x, x_n])$ is a group homomorphism with kernel $H \cap Z_1(G)$. Thus $(H \cap Z_2(G))/(H \cap Z_1(G))$ is finitely generated. Since $G$ is a linear group, $G/Z_1(G)$ is linear [5, Theorem 6.2] and we have that

$$(H \cap Z_1(G))/Z_1(G) \cong (H \cap Z_2(G))/(H \cap Z_1(G))$$

is finitely generated. By induction it follows that

$$HZ_1(G)/Z_1(G) \cong H/(H \cap Z_1(G))$$

is finitely generated and the result is clear.

Lemma 2. Let $G$ be a nilpotent linear group. Then the following are equivalent.

(i) If $x \in G$, the normal closure of $\langle x \rangle$ in $G$ is finitely generated.

(ii) $T_3(G)$ is finitely generated.

(iii) $G/Z_1(G)$ is finitely generated.

Proof. For arbitrary nilpotent groups (iii) implies (ii) [3, Corollary 3.19]. Trivially (ii) implies (i). Let $x_1, x_2, \ldots, x_n$ be elements of $G$ spanning $G$ linearly. If we suppose that the normal closure $F$ in $G$ of $\langle x_1, x_2, \ldots, x_n \rangle$ is finitely generated, the homomorphism

$$Z_2(G) \to F \times \cdots \times F$$

in which $x \mapsto ([x, x_1], \ldots, [x, x_n])$ proves that $Z_1(G/Z_1(G))$ is finitely generated. It follows from Lemma 1 that $G/Z_1(G)$ is finitely generated. This proves that (i) implies (iii).

Lemma 3. Let $G$ be a nilpotent group such that $G/Z_1(G)$ is finitely generated. Then $G$ is residually finite if and only if $Z_1(G)$ is residually finite.

Proof. It suffices to show that $R(Z_1(G)) = R(G)$. Trivially $R(Z_1(G)) \subseteq R(G)$. For if $N$ is a subgroup of $Z_1(G)$ of finite index, $N \triangleleft G$, so $G/N$ is finitely generated, nilpotent and hence residually finite [2, Theorem 2.1]. Thus $R(G) \subseteq N$ and $R(G) \subseteq R(Z_1(G))$.

We remark that Lemma 3 is a trivial consequence of [4, Proposition 1]. However, the above is quite sufficient for our purposes.

Proposition 4. Let $G$ be a residually linear nilpotent group satisfying the following conditions.

(i) If $x \in G$, the normal closure of $\langle x \rangle$ in $G$ is finitely generated.

(ii) $G/T_3(G)$ is residually finite and for each prime $p$ its $p$-torsion is bounded.

Then $G$ is residually finite.
PROOF. Let \( 1 \neq x \in G \). We will prove that \( x \not\in R(G) \). Since \( G \) is residually linear we can consider a homomorphism \( \phi \) of \( G \) into a linear group such that \( \phi(x) \neq 1 \). Let \( \bar{G} = G/(\text{Ker } \phi \cap \Gamma_2(G)) \). Since homomorphic images of \( G \) satisfy (i) it follows from Lemma 2 that \( \Gamma_2(G/\text{Ker } \phi) \) and \( (G/\text{Ker } \phi)/Z_1(G/\text{Ker } \phi) \) are finitely generated. Clearly \( \bar{G} \cong (G/\text{Ker } \phi) \times (G/\Gamma_2(G)) \). Then we see easily that \( \Gamma_2(\bar{G}) \) and \( \bar{G}/Z_1(\bar{G}) \) are finitely generated. Furthermore we have

\[
\bar{G}/\Gamma_2(\bar{G}) \cong G/\Gamma_2(G).
\]

Thus by (ii) we conclude that the \( p \)-torsion of \( \bar{G} \) is bounded for each prime \( p \). Therefore \( R(Z_1(\bar{G})) \) is a radicable group. But \( G/\Gamma_2(G) \) is residually finite so \( R(Z_1(\bar{G})) \subseteq \Gamma_2(\bar{G}) \). Since \( \Gamma_2(\bar{G}) \) is finitely generated, necessarily \( R(Z_1(\bar{G})) = \langle 1 \rangle \). Now Lemma 3 implies that \( \bar{G} \) is residually finite. Since \( x \not\in \text{Ker } \phi \cap \Gamma_2(G) \) we conclude that \( x \not\in R(G) \).

**Lemma 5.** Let \( G \) be a \( \mathcal{R} \) group and let \( H \) be a finitely generated subgroup. Then \( G/H \) is a \( \mathcal{R} \) group.

**Proof.** \( G \) is a \( \mathcal{R} \) group hence \( G \subseteq \mathbb{Z} \times C \). Where \( C \) is a bounded group and so a direct sum of cyclic groups [1, Theorem 17.2]. Since subgroups of \( \mathcal{R} \) groups are \( \mathcal{R} \) groups, we can assume \( G = \mathbb{Z} \times C \) in order to prove the lemma. Let \( H \) be a finitely generated subgroup of \( G \). Then there exist finitely generated subgroups \( M \subseteq \mathbb{Z} \) and \( N \subseteq C \) such that \( H \subseteq M \times N \). Every finitely generated subgroup of \( \mathbb{Z} \) can be embedded in a finitely generated direct summand of \( \mathbb{Z} \) [1, Theorem 19.2]. Clearly the same property holds for \( C \). Therefore we may, in addition, suppose that \( M \times M' = \mathbb{Z} \) and \( N \times N' = C \) for some \( M' \subseteq \mathbb{Z} \) and \( N' \subseteq C \). Now we have

\[
G/H \cong (M \times N/H) \times M' \times N'
\]

and the result follows.

We can now give the

**Proof of Theorem 1.** (i) By the proposition, we have only to prove that \( G/\Gamma_2(G) \) is residually finite and the elements of \( G/\Gamma_2(G) \) of finite order are of bounded order. We use induction on the class \( c \) of \( G \). If \( c = 1 \) the result is trivial. Suppose \( c > 1 \) and let \( \bar{G} = G/Z_1(G) \). Trivially \( \Gamma_2(\bar{G}) \) is finitely generated. \( Z_1(\bar{G}) \) is \( \mathcal{R} \), since it is contained in a product \( \mathbb{Z} \). By induction we have that \( \bar{G}/\Gamma_2(\bar{G}) \) is residually finite and its torsion is bounded. Since \( Z_1(G) \) is \( \mathcal{R} \) and \( \Gamma_2(G) \) is finitely generated it follows from Lemma 5 that \( Z_1(G)\Gamma_2(G)/\Gamma_2(G) \) is \( \mathcal{R} \). Therefore we have that \( G/\Gamma_2(G) \) is of torsion bounded. Thus \( R(G/\Gamma_2(G)) \) is a radicable group contained in \( Z_1(G)\Gamma_2(G)/\Gamma_2(G) \). Since \( \mathcal{R} \) groups contain no nontrivial radicable groups we conclude that \( G/\Gamma_2(G) \) is residually finite.

(ii) Let \( p \) be a prime. Put

\[
G = \langle z, x_i, y_i, \ i = 0, 1, \ldots : z_{i+1}^p = z_i, \ x_i, x_j = [y_i, y_j] = [z_i, z_j] = [z_i, x_j] = [z_i, y_j] = 1, \ [x_i, y_j] = 1 \text{ if } i \neq j, [x_i, y_j] = z_i^p \rangle.
\]
$G$ is a nilpotent group of class 2 with $\Gamma_2(G) = \langle z_0 \rangle$ and
\[ Z_1(G) = \langle z_i, i = 0, 1, \ldots \rangle \cong Q_p \]
(where $Q_p$ is the group of all rational numbers whose denominators are powers of $p$). $Z_1(G)$ is residually finite however it does not satisfy $\mathfrak{R}$. We will prove that $G$ is not residually finite but it is residually linear. Suppose that $x \mapsto \bar{x}$ is a homomorphism of $G$ into a finite group $\bar{G}$. Then, by the finiteness of $\bar{G}$, there exist distinct integers $n, m$ with $\bar{x}_n = \bar{x}_m$. Thus $\bar{1} = [\bar{x}_m, \bar{y}_m] = [\bar{x}_n, \bar{y}_n] = z_0^{pm}$. Since a finite homomorphic image of $Q_p$ has no elements of order $p$, we have that $z_0 = \bar{1}$ so $z_0 \in R(G)$. In fact $R(G) = Z_1(G)$. Let $K$ be a field containing the $p^n$-roots of the unity for any integer $n > 1$. In order to prove that $G$ is residually linear it suffices to show that the group $G_n = G/\langle z_0^n \rangle$ is residually $K$-linear for any integer $n > 1$, since $\cap_{n > 1} \langle z_0^n \rangle = \langle 1 \rangle$.

It follows from the relations of $G$ that $Z_1(G_n)$ is residually $p^n$-linear. Clearly $Z_1(G_n)$ has finite index in $G_n$ so $G_n$ is residually $K$-linear and the result follows.

(iii) Let $p$ be a prime. Let $G$ be a group generated by $z, t_1, x_i, y_i, i = 1, 2, \ldots$, subject to the relations
\[ x_i, x_j = [y_i, y_j] = [t_i, t_j] = [z, x_i] = [z, y_i] = 1, \]
\[ x_i, y_i = t_i^p z, \quad [x_i, y_j] = 1 \quad \text{if } i \neq j, \]
\[ t_i, x_i = [t_i, y_i] = z^{p^r}, \quad [t_i, y_j] = 1 \quad \text{if } i \neq j. \]

$G$ is a torsion free nilpotent group of class 3 with center $\langle z \rangle$. Let $x \mapsto \bar{x}$ be a homomorphism of $G$ into a finite group $\bar{G}$. Then, by finiteness of $\bar{G}$, there exist distinct integers $n, m$ with $\bar{y}_n = \bar{y}_m$. Thus $\bar{1} = [\bar{y}_n, \bar{y}_m] = [\bar{t}_n, \bar{y}_m]^m$. Suppose that $\bar{z} \neq \bar{1}$. Then $h_p(\bar{z})$, the $p$-height of $\bar{z}$ in $\bar{G}$, is finite. Again there exist distinct integers $r, s > h_p(\bar{z})$, with $1 = [\bar{x}_r, \bar{y}_r] = [\bar{x}_s, \bar{y}_s] = t_i^{p^r}$. Hence $\bar{z} = (t_i^{-1})^{p^r}$ and $s < h_p(\bar{z})$, a contradiction. Therefore we have shown that $z$ belongs to the kernels of all homomorphism of $G$ into finite groups so $z \in R(G)$. Finally we show that $G$ is residually $K$-linear, if $K$ contains, for every $n$, the $p^n$-roots of the unity. Define for each integer $n > 1$
\[ H_n = \langle t_1^{p^n}, t_2^{p^n}, \ldots, t_{n-1}^{p^n}, z^{p^n}, t_m^{p^n} z \rangle. \]

Clearly $H_n$ is a normal subgroup of $G$ and $H_n \cap \langle z \rangle = \langle z^{p^n} \rangle$. Therefore $\cap_{n > 1} H_n = \langle 1 \rangle$. Then it suffices to prove that the group $G = G/H_n$ is residually $K$-linear. It is clear that
\[ G = \langle Z_1(\bar{G}), \bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{y}_1, \ldots, \bar{y}_{n-1}, \bar{t}_1, \ldots, \bar{t}_{n-1} \rangle. \]

Furthermore, for $i = 1, 2, \ldots, n - 1$, we have
\[ t_i^{p^n} = \bar{1} \quad \text{so} \quad [\bar{x}_i^{p^n}, \bar{t}_i] = [\bar{y}_i^{p^n}, \bar{t}_i] = \bar{1}, \]
\[ [\bar{x}_i^{p^n}, \bar{y}_i] = [\bar{x}_i, \bar{y}_i]^{p^n} [\bar{x}_i, \bar{y}_i, \bar{x}_i]^{p^2} = t_i^{p^{n+1} z^{p^{n+1} (p^n - 1)/2}}. \]
similarly $[\tilde{y}^p, \tilde{x}] = \tilde{1}$. These relations yield that $\tilde{G}/Z_1(\tilde{G})$ is a torsion group which is finite, since it is finitely generated. The result follows, since $Z_1(\tilde{G})$ is residually $K$-linear.

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REFERENCES


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