A CHARACTERIZATION OF THE PEDERSEN IDEAL OF $C_0(T, B_0(H))$ AND A COUNTEREXAMPLE

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Abstract. Let $T$ be a locally compact Hausdorff space, $H$ a complex Hilbert space, and $A$ the $C^*$-algebra $C_0(T, B_0(H))$. Let $A_0$ be the Pedersen ideal of $A$ and $J_A$ the two-sided ideal of $A$ consisting of all $x$ having compact support, for which $\sup\{\dim x(t): t \in T\} < \infty$. It is known that $A_0 \subset J_A$, and equality has been conjectured by Pedersen. We give a new characterization of $A_0$ which enables us to show that the conjecture is false.

1. Introduction. Let $A$ be a $C^*$-algebra with continuous trace, $\hat{A}$ the spectrum of $A$, and $J_A$ the set of all $x$ in $A$ such that $\sup\{\dim \pi(x): \pi \in \hat{A}\} < \infty$ and $\pi(x) = 0$ for $\pi$ outside some compact subset of $\hat{A}$. In [2, 4.7.24, p. 100] Dixmier asked whether or not $J_A$ is the minimal dense two-sided ideal of $A$. Pedersen and Petersen answered this question negatively in [9, Proposition 3.6, p. 202]. By using homogeneous algebras whose corresponding fibre bundles have sufficiently many twists, Pedersen and Petersen were able to construct an example of a $C^*$-algebra $A$ with continuous trace for which $J_A$ is not the minimal dense two-sided ideal of $A$. In [8, p. 13] Pedersen did conjecture, however, that when $A = C_0(T, B_0(H))$, then $J_A$ is the minimal dense hereditary two-sided ideal of $A$, or equivalently, the minimal dense two-sided ideal (see [4, 2, p. 168]). Here $T$ is a locally compact Hausdorff space and $B_0(H)$ is the $C^*$-algebra of compact operators on some Hilbert space $H$. The minimal dense hereditary (order related) two-sided ideal of a $C^*$-algebra is commonly referred to as Pedersen’s ideal; this ideal was shown to exist in every $C^*$-algebra by Pedersen [6, 8].

In §2 of this note we give a new characterization of Pedersen’s ideal of $C_0(T, B_0(H))$. Consequently, in §3 we are able to construct an example that shows Pedersen’s conjecture is false. For basic concepts and definitions we refer the reader to [2], [6], [8].

2. Pedersen’s ideal of $C_0(T, B_0(H))$. Let $T$ be a locally compact Hausdorff space and $H$ a Hilbert space. Let $\mathcal{U} = \mathcal{U}(T)$ denote the set of all ordered triples $n = (U, \alpha, e)$ that satisfy the following:

(i) $U$ is an open subset of $T$;

(ii) $\alpha$ is a nonnegative continuous function defined on $T$ which has compact support and for which $\{t \in T: \alpha(t) > 0\} \subset U$;

(iii) $e$ is continuous mapping of $U$ into $H$ such that $\|e(t)\| = 1$ for all $t \in U$
(the topology for \( H \) is the norm topology).

For each \( n = (U, \alpha, e) \) define the map \( z_n: T \to B_0(H) \) by

\[
z_n(t) = \begin{cases} 
\alpha(t)P_e(t), & t \in U, \\
0 & \text{otherwise}, 
\end{cases}
\]

where \( P_e(t) \) denotes the projection of \( H \) onto \( H_t \), the subspace of \( H \) generated by \( e(t) \), and \( B_0(H) \) denotes the \( C^* \)-algebra of all compact operators on \( H \). Let \( C_0(T, B_0(H)) \) denote the \( C^* \)-algebra of all continuous maps \( x: T \to B_0(H) \) such that the real map \( t \to \|x(t)\| \) vanishes at infinity. Here the topology for \( B_0(H) \) is the norm topology. Finally, let \( A \) denote the \( C^* \)-algebra \( C_0(T, B_0(H)) \) and \( A_0 \) its Pedersen ideal.

2.1. **Lemma.** Let \( D = \{z_n: n \in \mathcal{N}\} \). Then the following statements hold:

(a) \( D \subseteq A^+ \);  
(b) \( D = \{z^{1/2}: z \in D\} \);  
(c) \( xDx^* \subseteq D \), for all \( x \in A \);  
(d) if \( 0 < x < z \), where \( x \in A \) and \( z \in D \), then \( x \in D \);  
(e) if \( u \in A \) and \( uu^* \in D \), then \( uu^* \in D \).

**Proof.** Clearly, (a), (b), and (d) hold. Now let \( x \in A \) and \( n = (U, \alpha, e) \in \mathcal{N} \). It is clear that the map \( t \to x(t)[e(t)] \) is continuous on \( U \); hence, \( V = \{t \in U: 0 < \|x(t)[e(t)]\|\} \) is an open subset of \( T \). Define

\[
f(t) = \left(1/\|x(t)[e(t)]\|\right)(x(t)[e(t)])
\]

for each \( t \in V \). Set

\[
\beta(t) = \begin{cases} 
\|x(t)[e(t)]\|^2\alpha(t), & t \in V, \\
0 & \text{otherwise}. 
\end{cases}
\]

Clearly, \( \beta(t) \) is a nonnegative continuous function defined on \( T \) with compact support and \( \{t \in T: \beta(t) > 0\} \subseteq V \). Now set \( m = (V, \beta, f) \), which certainly belongs to \( \mathcal{N} \). It is straightforward to show that \( xz_nx^* = z_m \). Hence (c) holds. Finally, suppose \( u \in A \) and \( uu^* = z \in D \). Then \( (uu^*)^2 = uu^* \in D \) by (c), hence \( uu^* \in D \) by (b). So (e) holds and our proof is complete.

2.2. **Theorem.** Let

\[
I = \left\{ \sum_{n \in \mathcal{F}} z_n: \mathcal{F} \subseteq \mathcal{N}, \mathcal{F} \text{ finite} \right\}.
\]

Then \( I \) is the minimal-dense, invariant order ideal (face) of \( A^+ \), that is, \( \text{span} \ I \) is the Pedersen ideal of \( A \).

**Proof.** Let \( x \in A^+ \) be so that \( x < \sum_{i=1}^p z_n \) for some finite subset \( n_1, n_2, \ldots, n_p \) of \( \mathcal{N} \). By the Riesz decomposition property [7, Corollary 2, p. 267], there are elements \( u_1, u_2, \ldots, u_p \) in \( A \) so that \( x = u_1u_1^* + \cdots + u_pu_p^* \) and \( u_i^*u_i < z_n, i = 1, 2, \ldots, p \). It follows from 2.1(d), (e) that \( x \in I \), so \( I \) is an order ideal (face) of \( A^+ \). Furthermore, by 2.1(c), \( I \) is an invariant order ideal of \( A^+ \) and by [2, 10.5.3, p. 199], \( \text{span} \ I \) is dense in \( A \). Thus \( I \) is a dense invariant order ideal, so \( A_0^+ \subseteq I \). To show \( I = A_0^+ \), it suffices to observe \( D \subseteq A_0 \). Let \( n = (U, \alpha, e) \in \mathcal{N} \) and choose \( h_0 \in H \) so that \( \|h_0\| = 1 \).

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Without loss of generality we may assume \( \|a^{1/2}\|_\infty < \frac{1}{2} \). Now set

\[
f(t) = \begin{cases} \frac{h_0 - a^{1/2}(t)e(t)}{\|h_0 - a^{1/2}(t)e(t)\|}, & t \in U, \\ h_0, & t \notin U, \end{cases}
\]

and

\[
g(t) = \begin{cases} \frac{h_0 + a^{1/2}(t)e(t)}{\|h_0 + a^{1/2}(t)e(t)\|}, & t \in U, \\ h_0, & t \notin U. \end{cases}
\]

Clearly, the maps \( t \mapsto f(t) \) and \( t \mapsto g(t) \) are continuous on all of \( T \). From [8, p. 8], we see that, for each \( \beta \in C_0(T)^+ \), \( \beta P_f \) and \( \beta P_g \) belong to \( A_0^+ \). So choose \( \beta \in C_0(T)^+ \) with \( \beta(t) = 1 \), \( t \in \text{supp} \alpha \), and \( \|\beta\|_\infty < 1 \). Now let \( h \in H \) and let \( t \in T \) be such that \( \alpha(t) > 0 \). Note

\[
\langle \alpha(t)P_e(t)[h], h \rangle = \alpha(t)|\langle h, e(t) \rangle|^2 
\]

\[
\leq 2\alpha(t)|\langle e(t), h \rangle|^2 + 2|\langle h, h_0 \rangle|^2 
\]

\[
= |\langle h, h_0 \rangle|^2 - 2\text{Re} \, \alpha^{1/2}(t)|\langle h, e(t) \rangle|\langle h_0, h \rangle + \alpha(t)|\langle e(t), h \rangle|^2 
\]

\[
+ |\langle h, h_0 \rangle|^2 + 2\text{Re} \, \alpha^{1/2}(t)|\langle h, e(t) \rangle|\langle h_0, h \rangle + \alpha(t)|\langle e(t), h \rangle|^2 
\]

\[
= \|h_0 - \alpha^{1/2}(t)e(t)\|^2|\langle h, f(t) \rangle|^2 
\]

\[
+ \|h_0 + \alpha^{1/2}(t)e(t)\|^2|\langle h, g(t) \rangle|^2 
\]

\[
< 4|\langle h, f(t) \rangle|^2 + 4|\langle h, g(t) \rangle|^2 
\]

\[
= 4\langle P_f(t)[h], h \rangle + 4\langle P_g(t)[h], h \rangle 
\]

\[
= 4\langle \beta(t)P_f(t)[h], h \rangle + 4\langle \beta(t)P_g(t)[h], h \rangle. 
\]

Thus \( z_n < 4\beta P_f + 4\beta P_g \). Since \( A_0^+ \) is an order ideal (face) of \( A^+ \), \( z_n \in A_0^+ \). So \( D \subseteq A_0 \) and our proof is complete.

3. Examples. We now detail the construction of a compact Hausdorff space \( T \) and an element \( x \) of the \( C^* \)-algebra \( A = C(T, B_0(H)) \) which does not belong to the Pedersen ideal of \( A \), even though each \( x(t) \) is a positive operator on \( H \) having dimension at most 1. The Hilbert space \( H \) is required to be infinite dimensional.

The building blocks for the space \( T \) are the complex projective spaces \( P^m \), which are defined as follows: \( P^m \) is the set of all 1-dimensional subspaces of \( C^{m+1} \), topologized as a quotient space of \( C^{m+1} \sim \{0\} \). The space \( P^m \) is a compact metric space. By identifying \( C^{m+1} \) with a fixed subspace of \( H \), we can view a point \( \pi \) of \( P^m \) as a 1-dimensional subspace of \( H \). To this subspace \( \pi \) we assign the projection operator \( x_m(\pi) \) which projects \( H \) onto \( \pi \). Since \( P_h \)
(the projection of $H$ onto the span of $h$) is continuous in $h$, for $h \in H \sim \{0\}$, and since $\chi_m(\pi) = P_h$ whenever $h \in \pi \sim \{0\}$, it follows that $\chi_m$ is a continuous function from $P^m$ to $B_0(H)$. Moreover, $\chi_m$ belongs to the Pedersen ideal of the $C^*$-algebra $C(P^m, B_0(H))$, because $\chi_m$ is positive and $\chi_m^2 = \chi_m$. The characterization of the Pedersen ideal given in the previous section applies to $\chi_m$ with the result that for some finite sequence $n(1), \ldots, n(k)$ in $\mathcal{P}(P^m)$,

$$\chi_m = \sum_{i=1}^k \chi_{n(i)}.$$

Let $\gamma(\chi_m)$ denote the smallest integer $k$ for which such a sequence $n(1), \ldots, n(k)$ exists. We will now prove that $\gamma(\chi_m) > m + 1$. This is the key to our example, and it is here that global topological properties of $P^m$ enter.

Let $\gamma_{m+1}$ be the canonical complex line bundle over $P^m$. The total space $E$ of $\gamma_{m+1}$ consists of all pairs $(\pi, \nu)$ such that $\pi \in P^m$ and $\nu \in \mathcal{P}$. The projection $p: E \to P^m$ is defined by $p(\pi, \nu) = \pi$. Suppose now that (1) holds with $n(i) = (U_i, \alpha_i, \nu_i)$, and let $V_i$ be the open subset of $U_i$ on which $\alpha_i$ is strictly positive. The sets $V_1, V_2, \ldots, V_k$ cover $P^m$ because $\chi_m$ is never zero. Since $\chi_m$ has rank 1 everywhere, it follows from (1) that if $\pi \in V_i$, then $\chi_m(\pi) = P_{\alpha_i}(\pi)$; or what amounts to the same thing, $\nu_i(\pi) \in \mathcal{P}$. We conclude that $(\pi, \nu_i(\pi)) \in E$ and $p(\pi, \nu_i(\pi)) = \pi$ whenever $\pi \in V_i$, which is precisely the statement that the bundle $\gamma_{m+1}$ admits a cross-section over $V_i$. Since this cross-section is never zero, $\gamma_{m+1}$ is trivial over $V_i$ [3, Exercise 1, p. 37]. Because each restriction $\gamma_{m+1}|V_i$ is trivial $(i = 1, 2, \ldots, k)$ there is a mapping $f: P^m \to P^{k-1}$ such that $\gamma_{m+1} \equiv f^*\gamma_1$, where $f^*\gamma_1$ is the induced bundle [3, Proposition 5.8, p. 31, and the proof of Theorem 5.5, p. 30]. The Chern class $c_1(\gamma_1)$ generates the integral cohomology ring $H^*(P^{k-1}, \mathbb{Z})$ and is carried by the induced cohomology homomorphism onto the Chern class $c_1(f^*\gamma_1)$ [3, pp. 232–233], [5, p. 160]:

$$f^*c_1(\gamma_1) = c_1(f^*\gamma_1) = c_1(\gamma_{m+1}).$$

We can conclude from (2) that $k > m$ because the $k$th power of $c_1(\gamma_1)$ is zero. This completes the proof that $\gamma(\chi_m) > m + 1$. (We are grateful to the referee for suggesting this proof.) We summarize our results in a theorem.

3.1. Theorem. Assume that $H$ is infinite dimensional. For each positive integer $m$, the $C^*$-algebra $C(P^m, B_0(H))$ contains an element $\chi_m$ such that $\chi_m(\pi)$ is a 1-dimensional projection for each $\pi \in P^m$, and for which $\gamma(\chi_m) > m + 1$.

Returning to the construction of our example, define $T$ to be the one-point compactification of the disjoint union of the $P^m$, $m = 1, 2, \ldots$:

$$T = \{\omega\} \cup \bigcup_{m=1}^{\infty} P^m.$$

Define an element $x$ of the $C^*$-algebra $A = C(T, B_0(H))$ by the formula
For each $t \in T$, $x(t)$ is positive and has dimension at most 1. However, $x$ cannot belong to the Pedersen ideal of $A$ because if it does, there must exist a finite sequence $n(1), \ldots, n(k)$ in $\mathcal{M}(T)$ such that

$$x = \sum_{i=1}^{k} z_{n(i)}$$

and by choosing an integer $m > k$ and restricting the terms of (3) to $P^m$, we obtain a sum of form (1) with $k < m$, contrary to Theorem 3.1. (When restricting the terms of (3) to $P^m$ we must also restrict the members of each triple $n(i)$ to $P^m$.) We state these results in the form of a theorem.

3.2. Theorem. Assume that $H$ is an infinite dimensional Hilbert space. There exists a compact metric space $T$ such that $C(T, B_0(H))$ contains a positive $x$ having dimension everywhere less than or equal to 1, which does not belong to the Pedersen ideal of $C(T, B_0(H))$.

It is worth pointing out that this example shows us the role of the mappings $e_i$ in our characterization of the Pedersen ideal. The example $x$ constructed above can be written in the form

$$x(t) = \begin{cases} m^{-1}x_m(t) & \text{if } t \in P^m, \\ 0 & \text{if } t = \omega. \end{cases}$$

where $P$ is a continuous projection valued map on $T \sim \{\omega\}$, and where $\alpha \in C(T)$.

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