NORM CONDITIONS ON RESOLVENTS OF SIMILARITIES
OF HILBERT SPACE OPERATORS AND APPLICATIONS
TO DIRECT SUMS AND INTEGRALS OF OPERATORS

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Abstract. Similarities of an operator $T$ are determined so that on certain
sets the norm of the resolvents of the similarity satisfy bounding conditions
independent of $T$. The results are applied to show that direct sums and
integrals of operators are quasi-similar to operators with spectrum depend-
ing only on the spectrum of the summands.

In this paper all operators will be bounded linear operators on Hilbert
space. Let $T$ be an operator and $A$ be a compact subset of the plane disjoint
from $\sigma(T)$, the spectrum of $T$. For any similarity $S$ of $T$ there are constants
$M_1$ and $M_2$ so that if $f$ is a function analytic on a neighborhood of $\sigma(T)$, then
$\|f(S)\| < M_1\|f(T)\|$ and $\|(S - \lambda I)^{-1}\| < M_2$ for all $\lambda \in A$. The main result
in this paper is to obtain similarities $S$ of $T$ so that the constants $M_1$ and $M_2$
are reasonably good. The best possible $M_2$ would be $M_0 = \text{dist}(A, \sigma(T))^{-1}$,
and, in fact, we show that any $M_2 > M_0$ is obtainable simultaneous with
$M_1 = 1$.

These resolvent growth results are used to show that direct sums and
integrals of operators are quasi-similar to operators with the smallest possible
spectrum. In particular, if $T = \Sigma \oplus T_i$, then it is always the case that
$\bigcup \sigma(T_i) \subset \sigma(T)$. However, we show that there is a quasi-similarity $S = \Sigma \oplus S_i$
of $T$, with $S_i$ similar to $T_i$ for each $i$ and $\sigma(S) = \bigcup \sigma(S_i) = \bigcup \sigma(T_i)$. We
wish to thank Larry Fiakalow for several helpful conversations and bringing
our attention to [2].

Whenever $\sigma(T)$ is contained in the interior of a disk of radius $r$, then there
exists a similarity $S$ of $T$ so that $\|(S - \lambda I)^{-1}\| < (|\lambda| - r)^{-1}$ whenever
$|\lambda| > r$. This follows from the well-known fact that the infimum of the norms
of similarities of $T$ is just the spectral radius of $T$ [3]. For $|\lambda| > r$ the norms of
$(S - \lambda I)^{-1}$ clearly do not depend on $T$ other than on the spectral radius of
$T$. The following two lemmas generalize this situation.

Lemma 1. Let $T$ be an operator with $\sigma(T)$ contained in the open unit disk.
There exists a similarity $S$ of $T$ so that $\|S\| \leq 1$ and $\|f(S)\| \leq \|f(T)\|$ for all
functions analytic on a neighborhood of $\sigma(T)$.

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**Proof.** Define $R = (\sum_0^\infty T^n x^n)^{1/2}$ and let $x \in H$. Then $\|Rx\|^2 = \langle R^2x, x \rangle = \langle \sum_0^\infty T^n x^n x, x \rangle$ so that

$$
\|Rx\|^2 = \sum_0^\infty \|T^n x\|^2.
$$

Hence $R$ is bounded above and below. Let $S = RTR^{-1}$; then by (1)

$$
\|RTR^{-1}x\|^2 = \sum_0^\infty \|T^{n+1}R^{-1}x\|^2
$$

$$
= \sum_0^\infty \|T^n R^{-1}x\|^2 - \|R^{-1}x\|^2
$$

$$
= \|x\|^2 - \|R^{-1}x\|^2.
$$

Therefore $\|S\| < 1$.

Now let $f$ be a function analytic on a neighborhood of $\sigma(T)$. It follows again from (1) that

$$
\|f(S)x\|^2 = \|Rf(T)R^{-1}x\|^2 = \sum_0^\infty \|T^n f(T)R^{-1}x\|^2
$$

$$
< \|f(T)\|^2 \sum_0^\infty \|T^n R^{-1}x\|^2.
$$

Hence, since $\sum_0^\infty \|T^n R^{-1}x\|^2 = \|x\|^2$, we obtain

$$
\|f(S)\| < \|f(T)\|. 
$$

**Remark.** In particular, $\|(S - \lambda I)^{-1}\| < \|(T - \lambda I)^{-1}\|$ for $\lambda$ not in $\sigma(T)$. By their definitions it follows that the operators $S$ and $R$ are in the $C^*$-algebra generated by $T$ and $I$.

If one simply manipulates an operator by scale, scalar change and inversion, then the preceding lemma can be used to get a bound on the resolvent of a similarity of $T$ near a point in the resolvent of $T$.

**Lemma 2.** Assume $\alpha$ is not in $\sigma(T)$ and fix an $r < d = \text{dist}(\alpha, \sigma(T))$. There exists a similarity $S$ of $T$ so that $\|(S - \lambda I)^{-1}\| < \|(T - \lambda I)^{-1}\|$ for $\lambda$ such that $|\lambda - \alpha| < r$ and for which $\|f(S)\| < \|f(T)\|$ for functions analytic on a neighborhood of $\sigma(T)$.

**Proof.** Take $T_0 = r(T - I)^{-1}$ to be the operator $T$ in Lemma 1. The operator $T_0$ satisfies the hypothesis of Lemma 1 so there exists an operator $S_0$ similar to $T_0$ such that $\|S_0\| < 1$ and $\|f(S_0)\| < \|f(T_0)\|$, whenever $f$ is analytic on a domain containing $\sigma(T_0)$.

If we let $S = rS_0^{-1} + \alpha I$, then $S$ is similar to $T$. Let $f$ be analytic on $\sigma(T)$ and $\psi(z) = (r + \alpha z)/z$. If $g = f \circ \psi$, then $g$ is analytic on $\sigma(S_0) = \sigma(T_0)$ so $\|g(S_0)\| < \|g(T_0)\|$. However $g(S_0) = f(S)$ and $g(T_0) = f(T)$ so $\|f(S)\| < \|f(T)\|$.

Finally the resolvent norm condition about $\alpha$ follows since $\|S_0\| < 1$. Specifically, $\|(S_0 - \gamma I)^{-1}\| < \|\gamma - 1\|^{-1}$, whenever $|\gamma| > 1$, thus
\[ \| (r(S - \alpha I)^{-1} - \gamma I)^{-1} \| \leq (|\gamma| - 1)^{-1}, \quad \text{for } |\gamma| > 1. \]

This yields \( \| (S - \alpha I)((1 + \mu \alpha)I - \mu S)^{-1} \| \leq r(|\mu| r - 1)^{-1}, \) for \( |\mu| > r^{-1}. \)

Hence
\[ \| (\mu^{-1}(1 + \mu \alpha)I - S)^{-1} \| \leq \| (S - \alpha)^{-1} \| |\mu| r (|\mu| r - 1)^{-1}, \]

for \( |\mu| > r^{-1}. \) However, \( r(S - \alpha)^{-1} = S_0 \) and \( \| S_0 \| < 1, \) so letting \( \delta = \mu^{-1}(1 + \mu \alpha) \) we obtain \( \| (\delta I - S)^{-1} \| \leq (r - |\delta - \alpha|)^{-1}, \) for \( |\delta - \alpha| < r. \)

Just as in Lemma 1, the operators in this lemma all belong to the \( C^* \)-algebra generated by \( T. \) By repeated application of Lemma 2 we obtain the following theorem. We use \( D^0 \) to be the interior of a set and \( \setminus D \) to be the complement of \( D. \)

**Theorem.** Let \( K, D \) be compact subsets of the plane so that \( K \subset D^0. \) For any constant \( M > (\text{dist}(K, \setminus D^0))^{-1} \) and for every operator \( T \) with \( \sigma(T) \subset K, \) there exists a similarity \( S \) of \( T \) so that for all functions \( f \) analytic on \( \sigma(T), \)

1. \( \| f(S) \| \leq \| f(T) \|, \)
2. \( \| (S - \lambda I)^{-1} \| \leq M \) for \( \lambda \notin D, \)
3. \( \| S \| \leq \sup_{\lambda \in D} |\lambda|. \)

**Proof.** Let \( T \) be any operator with \( \sigma(T) \subset K. \) Let \( d = \sup_{\lambda \in K} |\lambda| \) and \( \epsilon = \text{dist}(K, \setminus D^0). \) Choose \( \eta \) so that \( \epsilon > \eta > 0 \) and \( (\epsilon - \eta)^{-1} < M. \) Then by Lemma 1, there exists a similarity \( S \) of \( T \) satisfying the norm condition (1) for all possible \( f \) where \( \| S \| \leq d + \eta. \) Thus we may assume \( T \) has norm less than \( d + \eta. \) From here on the proof is somewhat messy, but obvious. For \( \lambda \notin D, \) let \( r(\lambda) = \text{dist}(\lambda, K). \) Let \( B_{\epsilon(\lambda)} \) denote the ball about \( \lambda \) of radius \( r(\lambda) \) and \( B \) the disk of radius \( d + \epsilon. \) Clearly for our result we may assume that \( B \supset D. \) Let \( \eta(\lambda) = r(\lambda) - \epsilon + \eta/2; \) then since each \( r(\lambda) > \epsilon \) it follows that \( \eta(\lambda) > \eta/2. \)

Consider the collection of sets \( \{ B_{\eta(\lambda)} \}, \) where \( \lambda \notin D. \)

By the compactness of \( B \setminus D^0, \) there are points \( \lambda_1, \ldots, \lambda_k \) for which \( \{ B_{\eta(\lambda)} \} \) covers \( B \setminus D^0. \) Notice that \( \epsilon \) and the \( \lambda \)'s depend only on \( K \) and \( D \) (\( \eta \) depends on our choice of \( M \)).

If we apply Lemma 2 to \( \lambda_1 \) and \( T, \) letting the \( r \) in Lemma 2 be \( r(\lambda_1) - \eta/2, \) we obtain a similarity \( S_1 \) of \( T \) with nice resolvent norm properties. Specifically if \( \lambda \in B_{\eta(\lambda)} \), then

1. \( \| (S_1 - \lambda I)^{-1} \| \leq (r(\lambda_1) - \eta/2 - \eta(\lambda_1))^{-1} \leq (\epsilon - \eta)^{-1} \leq M, \)

and

2. \( \| f(S_1) \| \leq \| f(T) \| \) for all possible \( f. \)

If we apply the above to \( \lambda_2 \) and \( S_1, \) we obtain an \( S_2 \) satisfying (1) on the set \( B_{\eta(\lambda_2)} \cup B_{\eta(\lambda_1)} \) and still satisfying \( \| f(S_2) \| \leq \| f(T) \| \) for all possible \( f. \) After \( k \)-steps we obtain \( S_k \) which satisfies

1. \( \| (S_k - \lambda I)^{-1} \| < M \) on \( B \setminus D^0, \) and
2. \( \| f(S_k) \| < \| f(T) \| \) for all possible \( f. \) To see that \( \| (S_k - \lambda I)^{-1} \| < M \) for \( \lambda \notin B \) recall that \( \| S_k \| \leq \| T \| \leq d + \eta \) and, thus.
Thus we have $S_k$ similar to $T$ with properties of the theorem satisfied.

A corollary of the theorem says that once you have obtained a bound on the complement of one compact set, going to a smaller set does not disturb that bound. This fact is a consequence of property (1) in the theorem.

**Corollary.** Let $\sigma(T) \subseteq K \subseteq D_1^0 \subseteq D_2^0$ where $D_1 \subseteq D_2$. If $M_i$ are constants given in the previous theorem for $D_i$ and $K$, then there exists a similarity $S$ of $T$ so that $\|(S - \lambda I)^{-1}\| < M_i$ if $\lambda$ is not in $D_i$ and $S$ satisfies the other conditions of the theorem.

The corollary can be used to obtain a result which was announced and independently obtained by D. Herrero [2].

**Proposition.** If $T = \sum \oplus T_n$ is a direct sum of operators, then $T$ is quasi-similar to an operator $S$ for which $\sigma(S) = \bigcup \sigma(T_n)$.

**Proof.** Let $K = \bigcup \sigma(T_n)$ and $D_n = K + 1/n$. Then there exists a similarity $S_n$ of $T_n$ so that $\|(S_n - \lambda I)^{-1}\| < M_i$ if $\lambda \notin D_i$ for $i = 1, \ldots, n$. Furthermore we may assume that if $d = \sup_{\lambda \in K} |\lambda|$, then $\|S_n\| < d + 1$. Let $S = \sum \oplus S_n$ and suppose $T_n = R_nS_nR_n^{-1}$. It easily follows that if $X = \sum \oplus R_n/\|R_n\|$ and $Y = \sum \oplus R_n^{-1}/\|R_n^{-1}\|$, then $XT = SX$ and $TY = YS$. Finally, by the growth conditions on the resolvents of $S_n$, it follows that for $\lambda \notin K$, $\|(S_n - \lambda I)^{-1}\|$ is uniformly bounded. Therefore, $\lambda \notin \sigma(S)$ and $\sigma(S) \subseteq K$.

The second application of the theorem involves direct integrals. For the details of direct integral decompositions, we refer to [4]; however, we shall introduce some basic notations and results here. Let $\mu$ be a finite positive regular measure defined on the Borel sets of a separable metric space $\Lambda$, and let $e_n$, $1 \leq n < \infty$, be a collection of disjoint Borel sets of $\Lambda$ with union $\Lambda$. Let $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_\infty$ be a sequence of Hilbert spaces, with $H_n$ having dimension $n$ and $H_\infty$ being separable. By

$$H = \int_\Lambda^\oplus H(t) \mu(dt)$$

we shall denote the space of weakly $\mu$-measurable functions from $\Lambda$ into $H_\infty$ such that $f(t) \in H_n$, if $t \in e_n$, and $\int_\Lambda \|f(t)\|^2\mu(dt) < \infty$. The space $H$ is a Hilbert space, and we shall denote the element $f \in H$ determined by the vector valued function $f(t)$ as $\int_\Lambda f(t) \mu(dt)$.

An operator $T$ on $H$ is said to be decomposable if there exists a $\mu$-measurable operator valued function $T(t)$ so that $(Tf)(i) = T(t)f(t)$ for $f \in H$. The operator $T$ is denoted by

$$T = \int_\Lambda^\oplus T(t) \mu(dt).$$

It is easy to see that $\lambda \notin \sigma(T)$ if and only if $\|(T(t) - \lambda I(t))^{-1}\|$ is essentially bounded. Thus the set $K = \cap \big\{ \bigcup_{t \in \delta} \sigma(T(t)) : \delta \text{ has full measure} \big\}$ is a compact subset of $\sigma(T)$. Since $K$ is the intersection of compact sets, there
exists a \( \delta \) of full measure so that \( K = \bigcup_{t \in \delta} \sigma(T(t)) \). Using the above theorem and proposition we can show that \( K \) is the spectrum of a quasi-similarity of \( T \).

**Proposition.** Let \( T \) be a decomposable operator and \( K \) as above. There exists a decomposable operator \( S \) which is quasi-similar to \( T \) and such that \( \sigma(S) = K \).

**Proof.** We shall show that \( T \) is the direct sum of operators each with spectrum in \( K \). Let

\[
g_{nm}(t) = \sup \left\{ \left( \frac{n}{\text{dist}(\lambda, K)} - \left\| (T(t) - \lambda I)^{-1} \right\| \right): \text{dist}(\lambda, K) > \frac{1}{m} \right\}
\]

and \( f_{nm}(t) = \min(0, g_{nm}(t)) \). From our theorem and corollary it follows that \( \lim g_{nm}(t) = 0 \) as \( n \to \infty \) for each \( m \) and \( t \). Choose \( \epsilon > 0 \). By Egoroff’s theorem there exists a set \( A_{me} \) so that \( f_{nm}(t) \to 0 \) uniformly as \( n \to \infty \) for \( t \) not in \( A_{me} \) and \( \mu(A_{me}) < \epsilon/2^m \). Let \( A_\epsilon = \bigcup A_{me} \); then \( \text{dist}(\lambda, K) > \frac{1}{m} \) uniformly as \( n \to \infty \) for all \( m \). Let \( B_\epsilon = \Lambda \setminus A_\epsilon \) and \( T_\epsilon = \int_{B_\epsilon} T(t) \mu(dt) \). It follows that \( \sigma(T_\epsilon) \subset K \) since \( \left\| (T(t) - \lambda I)^{-1} \right\| \) is uniformly bounded for \( t \) in \( B_\epsilon \) and \( \lambda \) not in \( K \).

By choosing \( \epsilon = 1/k \), and using an induction argument, it follows that \( T = \Sigma \bigoplus T_k \) with \( \sigma(T_k) \subset K \). Moreover, we have \( T_k = \int_{B_k} T(t) \mu(dt) \), where the \( \{B_k\} \) are disjoint measurable subsets of \( \Lambda \). It follows from the previous proposition that \( T = \Sigma \bigoplus T_k \) is quasi-similar to \( S = \Sigma \bigoplus S_k \), and \( \sigma(S) = \bigcup \sigma(S_k) = \bigcup \sigma(T_k) \) because \( S_k \) is similar to \( T_k \). From the proof of theorem it follows that \( S_k \) and the implementing similarities \( R_k \) belong to the von Neumann algebra generated by \( T_k \). Thus \( S_k = \int_{B_k} S_k(t) \mu(dt) \) with \( S_k(t) \) similar to \( T_k(t) = T(t) \) for \( t \) in \( B_k \) and, moreover, the operator \( R_k \) is also decomposable. Consequently, \( S = \Sigma \bigoplus S_k \) as well as the operators \( \Sigma \bigoplus R_k/\|R_k\| \) and \( \Sigma \bigoplus R_k^{-1}/\|R_k^{-1}\| \) which implement the quasi-similarity of \( S \) on \( T \) are all decomposable with respect to the given decomposition of \( H \).

**Remark.** Let \( \{M_n\} \) be a sequence of invariant subspaces of \( T \). C. Apostol calls \( M \) basic for \( T \) if for all \( n \) the subspaces \( M_n \) and \( \bigvee_{m \geq n} M_m \) are complementary and \( \bigcap_n \bigvee_{m \geq n} M_m = \{0\} \). If \( T_n = T/M_n \), then it is easy to show that \( T \) is quasi-similar to \( \Sigma \bigoplus T_n \) [1]. If \( \cup \sigma(T_n) = K \), then, as a corollary to the above results, we obtain that \( T \) is quasi-similar to an operator \( S \) with \( \sigma(S) \subset K \). In particular, if \( T/M_n \) is quasi-nilpotent for all \( n \), then \( T \) is quasi-similar to a quasi-nilpotent operator.

**References**


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