POSITION OF COMPACT HYPERSURFACES OF THE \( n \)-SPHERE

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Abstract. Let \( S^n \) be the Euclidean sphere of dimension \( n \). Let \( p \) and \( q \) be antipodal points on \( S^n \), and, for nonnegative \( h \), let \( C(p, h) \), \( C(q, h) \) be the hyperspheres of constant mean curvature \( h \) centered at \( p \) and \( q \), respectively. Then any closed hypersurface in \( S^n \) with mean curvature bounded by \( h \) must have a point in the 'tropical' region bounded by \( C(p, h) \) and \( C(q, h) \).

1. Introduction. Let \( S^n \) be the \( n \)-sphere with the standard Riemannian metric induced by inclusion in \( \mathbb{R}^{n+1} \). For \( p \in S^n \), let \( C(p, k) \) be the \((n - 1)\)-sphere of constant mean curvature \( k \), centered at \( p \). Let \( D(p, k) \) be the component of \( S^n \) \(-\ C(p, k) \) containing \( p \). We prove:

1.1 Theorem. Let \( M \) be a hypersurface in \( S^n \) which is smooth, compact, and without boundary. Let \( H \) be the mean curvature function on \( M \). If \( |H| < k \), then for any two antipodal points \( p \) and \( q \) in \( S^n \), there is a point of \( M \) lying in the set \( A(p, q, k) = S^n \) \(-\ (D(p, k) \cup D(q, k)) \).

Note that the boundary of \( A(p, q, k) \) is just \( C(p, k) \cup C(q, k) \). If \( M \) is minimal, then \( H \equiv 0 \), and we have

1.2 Corollary. Let \( M \) be a compact, oriented, minimal hypersurface without boundary in \( S^n \). Then \( M \) must intersect each great \((n - 1)\)-sphere.

The Corollary may also be proved using methods developed by H. B. Lawson [2].

2. Proof of the Theorem. We prove first the following

Lemma. If \( |H| < k \), then there is a point of \( M \) lying in \( S^n \) \(-\ D(p, k) \).

Proof. Suppose \( M \) lies entirely in \( D(p, k) \). Since \( M \) is compact, then it must also lie in the closure of \( D(p, r) \) with \( r > k \). Shrink \( D(p, r) \) until \( C(p, r) \) first touches \( M \). Then at some point \( m \), \( M \) is tangent to \( C(p, r) \). Since \( M \) is tangent from within \( D(p, r) \), it follows that \( H(m) > r > k \). But this contradicts \( |H| < k \).\(^2\)

The Theorem now follows by application of the Lemma to the antipodal points \( p \) and \( q \).

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